Paramodulation and Knuth-Bendix Completion with Nontotal and Nonmonotonic Orderings *

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Abstract. Up to now, all existing completeness results for ordered paramodulation and Knuth-Bendix completion require the term ordering $\succ$ to be well-founded, monotonic and total(izable) on ground terms. For several applications, these requirements are too strong, and hence weakening them has been a well-known research challenge.

Here we introduce a new completeness proof technique for ordered paramodulation where the only properties required on $\succ$ are well-foundedness and the subterm property. The technique is a relatively simple and elegant application of some fundamental results on the termination and confluence of ground term rewrite systems (TRS).

By a careful further analysis of our technique, we obtain the first Knuth-Bendix completion procedure that finds a convergent TRS for a given set of equations E and a (possibly non-totalizable) reduction ordering $\succ$ whenever it exists. Note that being a reduction ordering is the minimal possible requirement on $\succ$, since a TRS terminates if, and only if, it is contained in a reduction ordering.

Keywords: term rewriting, automated deduction.

1. Introduction

Deduction with equality is fundamental in mathematics, logics and many applications of formal methods in computer science. During the last two decades this field has importantly progressed through new Knuth-Bendix-like completion techniques and their extensions to ordered paramodulation for first-order clauses. These techniques have lead to important results on theorem proving in first-order logic with equality (Hsiang and Rusinowitch, 1991; Bachmair et al., 1986; Bachmair and Dershowitz, 1994; Bachmair and Ganzinger, 1994; Bachmair and Ganzinger, 1998; Nieuwenhuis and Rubio, 2001) (that have

* All authors partially supported by the ESPRIT Basic Research Action CCL-II, ref. WG # 22457. and the CICYT project HEMOSS ref. TIC98-0949-C02-01. A preliminary version of this work was presented at IEEE LICS'99.

been applied to state-of-the-art theorem provers like Spass (Weidenbach, 1997)), results on logic-based complexiy and decidability analysis (Basin and Ganzinger, 1996; Nieuwenhuis, 1998), on deduction with *constrained clauses* (Kirchner et al., 1990; Nieuwenhuis and Rubio, 1995), on inductive theorem proving (Comon and Nieuwenhuis, 2000), and on many other applications like symbolic constraint solving, or equational-logic programming.

But all previously known completeness results for Knuth-Bendix completion and ordered paramodulation require the term ordering $\succ$ to be well-founded, monotonic and total (or extendable to a total ordering) on ground terms. All main proof techniques, like the transfinite semantic tree method (Hsiang and Rusinowitch, 1991), the proof ordering method (Bachmair et al., 1986; Bachmair and Dershowitz, 1994), and the model generation method (Bachmair and Ganzinger, 1994) rely at some point on these requirements. Moreover, in many practical situations these requirements are too strong.

A typical situation is deduction modulo built-in equational theories E, where the existence of a total *E-compatible* reduction ordering is a very strong requirement. For example, the existence of such an ordering for the case where E consists of associativity and commutativity (AC) properties for some symbols remained open for a long time, and, once it was found, it triggered quite a number of results, like the decidability of the ground AC-word and -unification problems. Unfortunately, for many E such orderings cannot exist. For instance, when E contains an idempotency axiom $f(x,x) = x$, then if $s \succ t$, by monotonicity one should have $f(s,s) \succ f(s,t)$, which by E-compatibility implies $s \succ f(s,t)$ and hence non-well-foundedness.

In this article we introduce techniques for dropping the monotonicity requirement that, among other applications, open the door to deduction modulo many more classes of equational theories. The only properties required for $\succ$ are well-foundedness and the subterm property. This solves a well-known open problem (e.g. at the RTA list of open problems (RTA-LOOP, 2001) since 1995).

Our technique (given in Sections 3 to 7) is a variant of the model generation technique, with the main difference that the termination of the ground rewrite system R that defines the model is not a consequence of the ordering. Instead, termination of R follows from other properties. In one of the settings treated here, it follows from the irreducibility, w.r.t. R itself, of the right hand sides of all rules of R; in another setting, if R is contained in a well-founded ordering $\succ$ with the subterm property, its termination follows from the irreducibility of the right hand sides at non-topmost positions only. Since each terminating TRS R induces a reduction (i.e., well-founded, monotonic) ordering $\rightarrow_R^+$, we can then use induction on (an extension of) this reduction ordering for proving the main completeness results. These results are given here for paramodulation with general first-order clauses with eager *selection* of negative literals. In Section 10 we only briefly mention strategies with selection of positive literals.

Another well-known open problem (e.g. posed by N. Dershowitz and J-P. Jouannaud on the RTA list of open problems (RTA-LOOP, 2001) since its creation in 1991) that is solved here (in Section 8 of this article) concerns the Knuth-Bendix completion procedure. The aim of this procedure is to build a convergent rewrite system R for a given a set of equations E and an ordering $\succ$ on terms (see (Dershowitz and Jouannaud, 1990) for details). In fact, all current state-of-the-art theorem provers in pure equational logic, like Waldmeister (Hillenbrand et al., 1997), are based on variations of the Knuth-Bendix completion procedure. When given an ordering $\succ$ that can be extended to a total reduction ordering on ground terms, this procedure is complete in the sense that it always finds a (possibly infinite) convergent TRS R, logically equivalent to E, contained in $\succ$. This is very useful for automatically proving that an equation $s \simeq t$ follows from a set of equations E, because after a finite number of steps of such a procedure always a TRS R is reached by which a rewrite proof for $s \simeq t$ exists.

But a rewrite system already terminates if $\succ$ is only a reduction ordering (in fact, a rewrite system terminates if, and only if, it is contained in a reduction ordering). Hence a very natural (frequently asked) question is: what happens if we apply completion with a given desirable orientation by a reduction ordering that cannot be extended to a total one, like $f(a) \rightarrow f(b)$ and $g(b) \rightarrow g(a)$, for which a and b must be uncomparable in any monotonic extension? Here (in Section 8) this question is answered affirmatively: by a careful further analysis of our technique, we obtain the first practical Knuth-Bendix completion procedure that finds a convergent TRS for a given set of equations E and a (possibly non-totalizable) reduction ordering $\succ$ whenever it exists. Note that for arbitrary reduction orderings it does not always exist: in each E-congruence class there should be a single minimal element. For example, if $E = \{a \simeq b, a \simeq c\}$ then one of a, b or c should be smaller than the other two.

Finally, in Section 9 we shortly mention the applicability of techniques for *redundancy elimination* and *constraint inheritance* in the context of this article, and in Section 10 we discuss some directions for future work and give some counterexamples indicating the limitations for some of the extensions.

2. Basic Notions

We use the standard definitions of (Dershowitz and Jouannaud, 1990): $T(\mathcal{F}, \mathcal{X})$ ($T(\mathcal{F})$) is the set of (ground) terms over a set of symbols $\mathcal{F}$ and a denumerable set of variables $\mathcal{X}$ (over $\mathcal{F}$), the subterm of t at *position* p is denoted $t|_p$, the result of replacing $t|_p$ by s in t is denoted $t[s]_p$, and syntactic equality of terms is denoted by $\equiv$.

If $\rightarrow$ is a binary relation, then $\leftarrow$ is its inverse, $\leftrightarrow$ is its symmetric closure, $\rightarrow^+$ is its transitive closure and $\rightarrow^*$ is its reflexive-transitive closure. We write $s \rightarrow^! t$ if $s \rightarrow^* t$ and there is no t' such that $t \rightarrow t'$. Then t is called *irreducible* and a *normal form* of s (w.r.t. $\rightarrow$). The relation $\rightarrow$ is *well-founded* or *terminating* if there exists no infinite sequence $s_1 \rightarrow s_2 \rightarrow \ldots$ and it is *confluent* or *Church-Rosser* if the relation $\leftarrow^* \circ \rightarrow^*$ is contained in $\rightarrow^* \circ \leftarrow^*$. It is *locally confluent* if $\leftarrow \circ \rightarrow \; \subseteq \; \rightarrow^* \circ \leftarrow^*$. By Newman's lemma, terminating locally-confluent relations are confluent. A relation $\rightarrow$ on terms is *monotonic* if $s \rightarrow t$ implies $u[s]_p \rightarrow u[t]_p$ for all terms s, t and u and positions p. A *congruence* is a reflexive, symmetric, transitive and monotonic relation on terms.

An *equation* is a multiset $\{s, t\}$, denoted $s \simeq t$ or, equivalently, $t \simeq s$. A first-order clause is a pair of finite multisets of equations Γ (the *antecedent*) and Δ (the *succedent*), denoted by $\Gamma \rightarrow \Delta$. The empty clause $\Box$ is a clause where both Γ and Δ are empty.

A *rewrite rule* is an ordered pair of terms (s, t), written $s \rightarrow t$, and a set of rewrite rules R is a *term rewrite system* (TRS). The rewrite relation with R on $T(\mathcal{F}, \mathcal{X})$, denoted $\rightarrow_R$, is the smallest monotonic relation such that $l\sigma \rightarrow_R r\sigma$ for all $l \rightarrow r \in R$ and all substitution σ, and if $s \rightarrow_R t$ then we say that s *rewrites into* t with R. R is called terminating, confluent, etc. if $\rightarrow_R$ is. A rewrite system R is *convergent* if it is confluent and terminating; then every term t has a unique normal form w.r.t. $\rightarrow_R$, denoted by $nf_R(t)$, and then $s \simeq t$ is a logical consequence of R (where R is seen as a set of equations) iff $nf_R(s)$ is $nf_R(t)$.

Let R be a set of ground equations or rewrite rules. Then the congruence $\leftrightarrow_R^*$ defines an equality Herbrand interpretation denoted by R^*, where the only predicate $\simeq$ is interpreted by $s \simeq t$ iff $s \leftrightarrow_R^* t$. We write $s = t \in R^*$ if $s \leftrightarrow_R^* t$. R^* *satisfies* (is a *model* of) a ground clause $\Gamma \rightarrow \Delta$, denoted $R^* \models \Gamma \rightarrow \Delta$, if $R^* \not\supseteq \Gamma$ or $R^* \cap \Delta \neq \emptyset$. The empty clause $\Box$ is hence satisfied by no interpretation. R^* satisfies a set of clauses S, denoted by $R^* \models S$, if it satisfies every clause in S. For dealing with non-equality predicates, atoms A can be expressed by equations $A \simeq$ *true* where *true* is a new symbol.

3. Some properties of ground TRS and orderings

A (strict partial) *ordering* on $T(\mathcal{F}, \mathcal{X})$ is an irreflexive transitive relation $\succ$. It is a *reduction* ordering if it is well-founded and monotonic, and moreover, it is *stable under substitutions:* $s \succ t$ implies $s\sigma \succ t\sigma$ for all substitutions σ. It fulfils the *subterm property* if $\succ \supseteq \rhd$, where $\rhd$ denotes the strict subterm ordering.

DEFINITION 2. *A* west ordering *is a well-founded ordering on $T(\mathcal{F})$ that fulfils the subterm property and that is total on $T(\mathcal{F})$ (it is called* west *after well-founded, subterm and total).*

Not all well-founded orderings on terms can be extended to west orderings, even if they do not contradict the subterm property. For example, if $a \succ_1 f(b)$ and $b \succ_1 f(a)$, then, if $\succ$ is $(\succ_1 \cup \rhd)^+$, we get $a \succ f(b) \succ b \succ f(a) \succ a$. But every well-founded ordering can be totalized (Wechler, 1991), and hence every well-founded ordering satisfying the subterm property can be extended to a west ordering. We also have the following:

LEMMA 3. *Every reduction ordering $\succ_r$ can be extended to a west ordering.*

Proof: Let $\succ_{rs}$ be $(\succ_r \cup \rhd)^+$. Then $\succ_{rs}$ is well-founded. We will derive a contradiction from the existence of an infinite sequence $s_1 \succ_{rs} s_2 \succ_{rs} \ldots$ with s_1 minimal w.r.t. $\succ_r$. By monotonicity of $\succ_r$ we have that $s[t] \rhd t \succ_r u$ implies $s[t] \succ_r s[u] \rhd u$, i.e. the relations commute in this direction. Now since $\rhd$ is well-founded there should be some s_k in the sequence which is the first one such that $s_k \succ_r s_{k+1}$, and hence by applying the commutation property we can re-arrange the sequence obtaining an infinite sequence $s_1 \succ_r s'_2 \succ_{rs} \ldots s'_k \succ_{rs} s_{k+1} \succ_{rs} \ldots$ for some $s'_2, \ldots, s'_k$, which contradicts the minimality of s_1. $\square$

LEMMA 4. *Let R be a ground TRS such that for all rules $l \to r$ in R the term r is irreducible by R. Then R is terminating.*

Proof: Assume R is non-terminating. Then there exists an infinite rewrite sequence $t_1 \to_R t_2 \to_R \ldots$ It is easy to extract an infinite subsequence $s_1 \to_R s_2 \to_R \ldots$ of it where there is at least one rewrite step $s_i \to_R s_{i+1}$ at the topmost position, i.e., where $s_i \equiv l$ and $s_{i+1} \equiv r$ for some rule $l \to r$ in R. But then s_{i+1} is irreducible by R, contradicting the infiniteness assumption. $\square$

LEMMA 5. *Let $\succ$ be a west ordering, and let R be a ground TRS such that, for all $l \to r$ in R, $l \succ r$ and r is irreducible by R at non-topmost positions. Then R is terminating.*

Proof: If R is non-terminating there is an infinite sequence $s_1 \to_R s_2 \to_R \ldots$ with at least one rewrite step $s_i \to_R s_{i+1}$ at the topmost position, i.e., where $s_i \equiv l$ and $s_{i+1} \equiv r$ for some rule $l \to r$ in R. But then $s_{i+1} \to_R s_{i+2} \to_R \ldots$ is an infinite sequence with steps only at topmost positions, where $s_{i+1} \succ s_{i+2} \succ \ldots$ contradicting the well-foundedness of $\succ$. $\square$

Finally we will also use the following well-known results on orderings:

LEMMA 6 ((Dershowitz and Jouannaud, 1990)). *If R is a terminating TRS then $\to_R^+$ is a reduction ordering.*

Given an ordering $\succ$, we define $\succ_{mul}$ to be the smallest ordering on multisets such that: $M \cup \{s\} \succ_{mul} N \cup \{t_1, \ldots, t_n\}$ if $M = N$ and $s \succ t_i$ for all $i \in 1 \ldots n$. If $\succ$ is well founded and total on S, so is $\succ_{mul}$ on finite multisets over S (Dershowitz and Manna, 1979).

4. West orderings in practice

In practical applications (theorem provers, implementations of Knuth-Bendix completion) the west ordering $\succ$ can be defined and dealt with in different ways.

One possibility is that an approximation of $\succ$ is available by a non-total ordering $\succ_v$ on terms with variables such that $s \succ_v t$ implies $s\sigma \succ t\sigma$ for all ground σ (e.g., when $\succ_v$ is a reduction ordering of which $\succ$ is an extension, as in Lemma 3). Then, inferences with ordering restrictions like $l\sigma\theta \succ r\sigma\theta$ for some ground substitution θ can be proved redundant by showing that $r\sigma \succeq_v l\sigma$. Indeed, the actual west ordering need not always be really built for application purposes. It suffices for completeness that it exists, and for practice that a reasonably good approximation is available.

We now mention two general-purpose techniques for defining $\succ$ such that in practice it can be used or approximated efficiently.

4.1. Semantic path orderings

The *recursive path ordering* with status (RPO, (Dershowitz, 1982)) (which includes the *lexicographic path ordering*, LPO), is a well-known, easy to implement, general-purpose ordering for deduction purposes. It is a reduction ordering that is total on ground terms. It compares the head symbols of the terms (with a *precedence* ordering on the function symbols), and then recursively it applies a (sometimes lexicographic or multiset) comparison on the arguments. Since RPO is monotonic and includes the subterm relation, it cannot prove termination of rules like $f(f(x)) \to f(g(f(x)))$, because the term $g(f(x))$ is larger than its subterm $f(x)$, and hence, by monotonicity, $f(g(f(x)))$ will always be larger than $f(f(x))$.

The Semantic Path Ordering (SPO, (Kamin and Levy, 1980)) is a well-known powerful generalization of the RPO, where the precedence on function symbols is replaced by any (well-founded) underlying (quasi-)ordering involving the whole term rather than only its head symbol. This makes the ordering much more powerful. In fact, for every terminating TRS R there is some SPO that includes $\to_R^+$. SPO includes the subterm relation, but it is not monotonic in general. In fact, it can handle rules like $f(f(x)) \to f(g(f(x)))$. The price to be paid is that for proving termination of TRS by an SPO, one needs to prove in an ad-hoc way its monotonicity for contexts of rule instances, that is, for all terms s and t such that s rewrites to t by R in one step.

Since the results of this paper imply that for deduction by ordered paramodulation monotonicity is not needed, and any SPO can be extended to a west ordering, SPO is an interesting candidate for these purposes. Furthermore, in Section 8 we will show that if R is a convergent TRS for some set of equations E, and $\to_R^+$ is included in a west ordering (like an SPO), then unfailing Knuth-Bendix completion on E with this west ordering will compute this R.

4.2. Non-monotonic E-compatible orderings

As mentioned in the introduction, an important bottleneck for defining deduction techniques modulo built-in equational theories E is to find the required total (up to E-equal ground terms), *E-compatible* reduction ordering. But if the monotonicity requirement can be dropped, this becomes a much simpler task.

Let us consider as an example such an ordering for the case where E consists of associativity and commutativity (AC) properties. The AC-compatibility of such an ordering $\succ$ means that $s =_{\mathrm{AC}} s' \succ t =_{\mathrm{AC}} t'$ implies $s' \succ t'$. It is easy to check whether two ground terms are AC-equal by using their *flattened* forms: a term s can be *flattened* by removing all AC-operators f that are immediately below another f. For example, if f and g are AC-operators, then the term $h(f(f(a,a), f(b, g(c, g(d,e)))))$ is flattened into $h(f(a,a,b,g(c,d,e)))$. Two terms are AC-equal if, and only if, their flattened forms are equal up to permutation of arguments of AC-operators.

Therefore, when trying to define total AC-compatible orderings, the first idea that comes to mind is to apply general-purpose orderings on the flattened forms of the terms to be compared, like an RPO where

AC-symbols have multiset status. But the resulting ordering may not be monotonic: if f is an AC-operator that is larger than g, then $f(a,a)$ will be larger than $g(a,a)$, but $f(g(a,a),a)$ will be larger than $f(a,f(a,a))$, because the latter term becomes $f(a,a,a)$ after flattening. However, it is not difficult to see that one obtains an AC-compatible west ordering from this simple approach.

5. Paramodulation with equations

In this section we introduce part of our ideas for the purely equational case. Dealing with this simple case first is useful not only for explanation purposes, but also because its results will be used in Section 8 on Knuth-Bendix completion. In the following, let $\succ$ be a given west ordering.

DEFINITION 7. *The inference rule of (equational)* ordered paramodulation *with respect to* $\succ$ *is:*

$$\frac{l \simeq r \qquad s \simeq t}{(s[r]_p \simeq t)\sigma}$$

where $\sigma = mgu(s|_p, l)$, the most general unifier of $s|_p$ and l, where $s|_p$ is not a variable, and l is maximal in its premise, that is, for some ground substitution θ, it holds that $l\sigma\theta \succ r\sigma\theta$.

As said, the usefulness of the ordering restrictions in practical applications depends on the way the west ordering $\succ$ is given. For example, if only an approximation is given by a non-total ordering $\succ_v$ on terms with variables such that $s \succ_v t$ implies $s\sigma \succ t\sigma$ for all ground σ, then the inference is not needed if, for instance, $r\sigma \succeq_v l\sigma$.

We now define, by induction on $\succ_{mul}$, a ground TRS R_E *generated* by a set of equations E:

DEFINITION 8. *Let E be a set of equations. An instance e of the form $l \simeq r$ of an equation in E* generates *the rule $l \rightarrow r$ if*

1. *$l \succ r$, and*
2. *l and r are irreducible by R_e*

where R_e is the set of rules generated by all instances d of equations in E such that $e \succ_{mul} d$. We denote by R_E the set of rules generated by all ground instances of E.

The previous construction is similar to the one of (Bachmair and Ganzinger, 1994), but here the rules are oriented only by a (non-monotonic) west ordering. Hence the termination of R_E has to be ensured otherwise, and therefore we require not only the left hand sides to be irreducible, but also the right hand sides.

PROPERTY 9. *Let E be a set of equations. Then for all rules $l \to r$ in R_E we have*

1. *r is irreducible by R_E*
2. *l is irreducible by $R_E \setminus \{l \to r\}$*

Proof: For the first property, by construction, if an instance e generates $l \to r$, the term r is irreducible by R_e. Since $l \succ r$, and $\succ$ is irreflexive and fulfils the subterm property, clearly $l \to r$ itself does not reduce r either. Finally, for every rule $l' \to r'$ generated by an instance d with $d \succ_{mul} e$, we must have $l' \succeq l$ and hence $l' \succ r$ which implies that l' cannot be a subterm of r either.

For the second property, by construction, if an instance e generates $l \to r$, then the term l is irreducible by R_e. Now, as before, for every rule $l' \to r'$ generated by an instance d with $d \succ_{mul} e$ we must have $l' \succeq l$. Since l' must be irreducible by R_d which contains $l \to r$ we have $l' \not\equiv l$, and hence $l' \succ l$, which implies that l' cannot be a subterm of l. □

LEMMA 10. *For every set of equations E, the ground TRS R_E is convergent.*

Proof: Termination follows by Property 9.1 and Lemma 4. For confluence, since R_E is terminating, by Newman's lemma we only need to show that R_E is locally confluent, which holds by Property 9.2. □

THEOREM 11. *Let E be a set of equations closed under ordered paramodulation with respect to a west ordering $\succ$. Then $R_E^* \models E$.*

Proof: Since R_E is terminating, by Lemma 6 it follows that $\to_{R_E}^+$ is a reduction ordering. We now proceed by induction on the well-founded ordering on ground equations $(\to_{R_E}^+)_{mul}$, denoted in the remainder of this proof by $\succ_R$. A contradiction is derived from the existence of a minimal w.r.t. $\succ_R$ ground instance e of the form $s\sigma \simeq t\sigma$ of an equation $s \simeq t$ in E such that $R_E^* \not\models e$. Since $R_E^* \not\models e$, the equation e is not a tautology of the form $u \simeq u$, and hence one its sides is strictly larger w.r.t. the (total) ordering $\succ$ than the other one. Moreover, again since

$R_E^* \not\models e$, the equation e has not generated any rule of R_E. This must be because either $s\sigma$ or $t\sigma$ is reducible by R_e. Then there exists some equation $l \simeq r$ that has generated a rule $l\sigma \rightarrow r\sigma$ reducing $s\sigma$ or $t\sigma$. We consider the case where $s\sigma$ is reducible; the other one is analogous. Now we have $s\sigma|_p \equiv l\sigma$, and there are two possibilities:
(i) An inference: $s|_p$ is a non-variable position of s.
Then there exists an inference by ordered paramodulation:

$$\frac{l \simeq r \qquad s \simeq t}{(s[r]_p \simeq t)\theta}$$

whose conclusion has an instance d of the form $(s[r]_p \simeq t)\sigma$, such that $e \succ_R d$ and $R_E^* \not\models d$, contradicting the minimality of e. Note that this inference satisfies the ordering constraints of ordered paramodulation, since $l\sigma \succ r\sigma$.
(ii) Lifting: $s|_{p'}$ is a variable x for some prefix p' of p.
Then $p = p' \cdot p''$ and $x\sigma|_{p''}$ is $l\sigma$. Now let σ' be the ground substitution with the same domain as σ but where $x\sigma' \equiv x\sigma[r\sigma]_{p''}$ and $y\sigma' \equiv y\sigma$ for all other variables y. Then $R_E^* \not\models s\sigma' \equiv t\sigma'$ and $e\sigma \succ_R e\sigma'$, contradicting the minimality of e. □

5.1. A slightly stronger paramodulation rule

In the proof of termination of R_E (lemma 10) we can as well use lemma 5 instead of lemma 4. More precisely, we only need the right hand sides of the rules in R_E to be irreducible at non-topmost positions (instead of being completely irreducible). Due to this observation, we can restrict our paramodulation rule avoiding its application at topmost positions of small sides of equations. Then we obtain the following *strict ordered paramodulation* rule:

DEFINITION 12. *The inference rule of (equational)* strict ordered paramodulation *with respect to $\succ$ is:*

$$\frac{l \simeq r \qquad s \simeq t}{(s[r]_p \simeq t)\sigma}$$

where $\sigma = mgu(s|_p, l)$, the most general unifier of $s|_p$ and l, where $s|_p$ is not a variable, and l is maximal in its premise, and if $p = \lambda$ then s is also maximal in its premise, that is, for some ground substitution θ, it holds that $l\sigma\theta \succ r\sigma\theta$, and, if $p = \lambda$ then we also have $s\sigma\theta \succ t\sigma\theta$.

Now to prove theorem 11, we slightly modify Definition 8, the generation of R_E, obtaining the following variant of it:

DEFINITION 13. *Let E be a set of equations. An instance e of the form $l \simeq r$ of an equation in E* generates *the rule $l \to r$ if*

1. *$l \succ r$,*
2. *l is irreducible by R_e, and*
3. *r is irreducible by R_e at non-topmost positions.*

where R_e is the set of rules generated by all instances d of equations in E such that $e \succ_{mul} d$. In the remainder of this section, we denote by R_E the set of rules generated by all ground instances of E.

We can now adapt Property 9, and prove it analogously. From this property, lemma 10 holds as before but using lemma 5 to conclude termination.

PROPERTY 14. *Let E be a set of equations. Then for all rules $l \to r$ in R_E we have*

1. *r is irreducible by R_E at non-topmost positions.*
2. *l is irreducible by $R_E \setminus \{l \to r\}$.*

Now we adapt the proof of Theorem 11 for the strict ordered paramodulation rule.

THEOREM 15. *Let E be a set of equations closed under strict ordered paramodulation with respect to a west ordering $\succ$. Then $R_E^* \models E$.*

Proof: As in Theorem 11, we proceed by induction on the well-founded ordering on ground equations $\succ_R$, deriving a contradiction from the existence of a minimal w.r.t. $\succ_R$ ground instance e of the form $s\sigma \simeq t\sigma$ of an equation $s \simeq t$ in E such that $R_E^* \not\models e$. Since $s\sigma$ cannot be equal to $t\sigma$, we assume w.l.o.g. that $s\sigma \succ t\sigma$.

The equation e has not generated any rule because $s\sigma$ is reducible by R_e or $t\sigma$ is reducible by R_e at a non-topmost position. Then there exists some equation $l \simeq r$ that has generated a rule $l\sigma \to r\sigma$, which reduces $s\sigma$ or $t\sigma$. The case where $s\sigma$ is reducible is analogous as in Theorem 11. For the other case we have $t\sigma|_p \equiv l\sigma$, for some position $p \neq \lambda$. If p is below a variable position in t then we apply the same lifting argument as in Theorem 11, and otherwise, since $p \neq \lambda$, we can conclude as well by the existence of an ordered paramodulation inference with a smaller conclusion. $\square$

This strict ordered paramodulation rule of Definition 12 will be adapted for the Horn case and the case of general clauses in Sections 6 and 7 respectively, and the non-strict one of Definition 7 will be used to obtain the results on unfailing Knuth-Bendix completion in Section 8.

6. The Horn case

In this section we generalise the results of the previous section to Horn clauses. In the following inference system it is assumed that in each clause with a non-empty antecedent one of these negative equations, the one that is written underlined, has been *selected* (see, e.g., (Bachmair and Ganzinger, 1998)). In the Horn case this leads to positive unit strategies (and in the non-Horn case to positive strategies): left premises of paramodulations are unit clauses, and the only inferences involving non-unit clauses are equality resolution or paramodulation left on its selected equation.

DEFINITION 16. *The inference system $\mathcal{H}$ for Horn clauses with respect to the west ordering $\succ$ is defined as follows:*

paramodulation right:

$$\frac{\to l \simeq r \qquad \to s \simeq t}{\to (s[r]_p \simeq t)\sigma} \qquad \text{where } \sigma = mgu(l, s|_p)$$

paramodulation left:

$$\frac{\to l \simeq r \qquad \Gamma, \underline{s \simeq t} \to \Delta}{(\Gamma, s[r]_p \simeq t \to \Delta)\sigma} \qquad \text{where } \sigma = mgu(l, s|_p)$$

equality resolution:

$$\frac{\Gamma, \underline{s \simeq t} \to \Delta}{(\Gamma \to \Delta)\sigma} \qquad \text{where } \sigma = mgu(s, t)$$

where moreover in both paramodulation rules $s|_p$ is not a variable, l is maximal in its premise, and if $p = \lambda$ then s is also maximal in its premise, that is, for some ground substitution θ, it holds that $l\sigma\theta \succ r\sigma\theta$, and, if $p = \lambda$ then also $s\sigma\theta \succ t\sigma\theta$.

DEFINITION 17. *Let S be a set of Horn clauses. We denote by E_S the subset of all positive unit clauses in S, that is, $E_S = \{\ s \simeq t \ \mid \to s \simeq t \in S\ \}$, and we denote by R_S the set of rules generated by E_S as in Definition 13 (i.e. $R_S = R_{E_S}$).*

We now use multiset extensions for lifting orderings $\succ$ on terms to orderings on equations and clauses. Let C be a ground clause, and let $emul(s \simeq t)$ be $\{s, t\}$ if $s \simeq t$ is a positive equation in C, and

$\{s, s, t, t\}$ if it is negative. Then, if $\succ$ is an ordering, we define the ordering $\succ_e$ on (occurrences of) ground equations in a clause by $e \succ_e e'$ if $emul(e) \succ_{mul} emul(e')$. Similarly, $\succ_c$ on ground clauses is defined $C \succ_c D$ if $mse(C)\ (\succ_{mul})_{mul}\ mse(D)$, where $mse(C)$ is the multiset of all $emul(e)$ for ocurrences e of equations in C.

THEOREM 18. *(refutation completeness of $\mathcal{H}$ for Horn clauses)* *Let S be a set of Horn clauses closed under $\mathcal{H}$. Then $\Box \in S$ if, and only if, S is unsatisfiable.*

Proof: The left to right implication is trivial. For the other one, let S be a set of clauses closed under $\mathcal{H}$ such that $\Box \notin S$. As in Theorem 11, we proceed by induction on $\succ_c^{R_S}$, which will be denoted in the remainder of this proof by $\succ_R$. This ordering is monotonic where needed (see the cases below) and well-founded. Again it is proved that R_S^* is a model of S by deriving a contradiction from the existence of a minimal w.r.t. $\succ_R$ ground instance C of a clause in S such that $R_S^* \not\models C$. Since E_S is closed under strict ordered paramodulation, by Theorem 15, we have $R_S^* \models E_S$, and therefore C cannot be a positive unit clause. Two cases have to be considered:

1. C is an instance $\Gamma\sigma, \underline{s\sigma \simeq t\sigma} \rightarrow \Delta\sigma$ of a clause $\Gamma, \underline{s \simeq t} \rightarrow \Delta$, where $s\sigma \equiv t\sigma$. Then there is an inference by equality resolution

$$\frac{\Gamma, \underline{s \simeq t} \rightarrow \Delta}{(\Gamma \rightarrow \Delta)\theta}$$

   whose conclusion has an instance D of the form $\Gamma\sigma \rightarrow \Delta\sigma$ such that $C \succ_R D$ and moreover, D is in S and $R_S^* \not\models D$, which is a contradiction.

2. C is an instance $\Gamma\sigma, \underline{s\sigma \simeq t\sigma} \rightarrow \Delta\sigma$ of a clause $\Gamma, \underline{s \simeq t} \rightarrow \Delta$, where, w.l.o.g., $s\sigma \succ t\sigma$.

   Then, since $R_S^* \not\models C$, we have $R_S^* \models s\sigma \simeq t\sigma$, and, since R_S is convergent, there must be a rewrite proof of $s\sigma \simeq t\sigma$ by R_S, that is, $s\sigma$ and $t\sigma$ must rewrite into the same normal form by R_S. This implies that either $t\sigma$ is reducible at a non-topmost position or else $s\sigma$ is reducible. (Note that it cannot be the case that the only possible reduction step on $s\sigma \simeq t\sigma$ is at the topmost position of $t\sigma$. By such a step, a new term t' is obtained with $t\sigma \succ t'$ and t' again irreducible at non-topmost positions; since $s\sigma \succ t\sigma$, such a sequence of topmost steps on $t\sigma$ can never produce $s\sigma$).

   Then either the lifting argument applies like in Theorem 11, or else there is an inference by paramodulation left:

$$\frac{\rightarrow l \simeq r \qquad \Gamma, \underline{s \simeq t} \rightarrow \Delta}{(\Gamma, s[r]_p \simeq t \rightarrow \Delta)\theta}$$

whose conclusion has an instance D of the form $(\Gamma, s[r]_p \simeq t \to \Delta)\sigma$ such that $C \succ_R D$ and $R_S^* \not\models D$, which is a contradiction. $\square$

7. General Clauses

Now we consider general clauses. As for the Horn case, we consider that in each clause with a non-empty antecedent one of its negative equations, the one that is written underlined, has been selected.

DEFINITION 19. *The inference system $\mathcal{I}$ with respect to the given west ordering $\succ$ is defined as follows:*

paramodulation right:

$$\frac{\to l \simeq r, \Delta \qquad \to s \simeq t, \Delta'}{(\to s[r]_p \simeq t, \Delta, \Delta')\sigma} \qquad \textit{where } \sigma = mgu(l, s|_p)$$

paramodulation left:

$$\frac{\to l \simeq r, \Delta \qquad \Gamma, \underline{s \simeq t} \to \Delta'}{(\Gamma, s[r]_p \simeq t \to \Delta, \Delta')\sigma} \qquad \textit{where } \sigma = mgu(l, s|_p)$$

equality resolution:

$$\frac{\Gamma, \underline{s \simeq t} \to \Delta}{(\Gamma \to \Delta)\sigma} \qquad \textit{where } \sigma = mgu(s, t)$$

equality factoring:

$$\frac{\to s \simeq t, s' \simeq t', \Delta}{(t \simeq t' \to s \simeq t, \Delta)\sigma} \qquad \textit{where } \sigma = mgu(s, s')$$

where in both paramodulation rules $s|_p$ is not a variable. The ordering restrictions are as follows:

In paramodulation right, l is strictly maximal in $l \simeq r$, and $l \simeq r$ is the strictly maximal equation in its premise, and if $p = \lambda$, then s is strictly maximal in $s \simeq t$ and $s \simeq t$ is the strictly maximal equation in its premise. Formally: for some ground substitution θ it holds that $l\sigma\theta \succ r\sigma\theta$ and $(l \simeq r)\sigma\theta \succ_e e\sigma\theta$ for all equations e in Δ, and if $p = \lambda$, then $s\sigma\theta \succ t\sigma\theta$ and $(s \simeq t)\sigma\theta \succ_e e\sigma\theta$ for all equations e in Δ'.

In paramodulation left, l is strictly maximal in $l \simeq r$, and $l \simeq r$ is strictly maximal in its premise, and if $p = \lambda$, then s is strictly maximal in $s \simeq t$.

Finally, in equality factoring, s is strictly maximal in $s \simeq t$ and $s \simeq t$ is maximal.

DEFINITION 20. *Let S be a set of clauses. An instance C of the form $\to l \simeq r, \Delta$ of a clause in S* generates *the rule $l \to r$ if*

1. $R_C^* \not\models C$,
2. $l \succ r$, *and* $(l \simeq r) \succ_e e$ *for all equations e in Δ,*
3. *l is irreducible by R_C,*
4. *r and Δ are irreducible at non-topmost positions by R_C*
5. $R_C^* \models r \simeq t$ *for no equation $l \simeq t$ in Δ.*

where R_C is the set of rules generated by all instances D of clauses in S such that $C \succ_c D$. In the remainder of this section, we denote by R_S the set of rules generated by all ground instances of S.

Again we have the following property which implies convergence of R_S.

PROPERTY 21.
Let S be a set of clauses. Then for all rules $l \to r$ in R_S we have

1. *r is irreducible by R_S at non-topmost positions.*
2. *l is irreducible by $R_S \setminus \{l \to r\}$.*

LEMMA 22. *Let S be a set of clauses. Then the ground TRS R_S is convergent.*

LEMMA 23. *Let S be a set of clauses. If $\to l \simeq r, \Delta$ is an instance C of a clause in S that generates the rule $l \to r$ in R_S, then $R_S^* \not\models \Delta$.*

Proof: We first prove that if $R_S^* \models s \simeq t$ for ground s and t that are irreducible at non-topmost positions and such that $s \succ t$, then no rules with left hand sides greater than s (w.r.t. $\succ$) are used in the rewrite proof. We proceed by induction on the size of $s \simeq t$ w.r.t. $\succ_{mul}$. The first step can only apply at topmost position of s or t. If it is on t then we obtain $s \simeq r$, where $t \to r$ is the applied rule in R_S, and hence $s \succ t \succ r$, by definition of R_S, and r is irreducible at non-topmost positions by Property 21. 1, and then we can conclude by induction hypotheses. Otherwise, we obtain $r \simeq t$, where $s \to r$ is the applied rule in R_S, and hence as before $s \succ r$ and r is irreducible at non-topmost positions. If $r \equiv t$ we are done, and otherwise if $r \succ t$ or $t \succ r$ we can conclude by induction.

Now, assume $s \simeq t$ is a ground equation in Δ with a rewrite proof using R_S and such that $s \succ t$. The rules generated by clauses D with $D \succ_c C$ cannot be used, since they have left hand sides greater than l

and hence greater than s w.r.t. $\succ$, and s and t are irreducible at non-topmost positions. But since $R_C^* \not\models s \simeq t$ the rule $l \to r$ is used. Hence $s \equiv l$ and $s \simeq t$ rewrites into $r \simeq t$, to which $l \to r$ cannot be applied any more, which contradicts the last condition of Definition 20. $\square$

We now introduce a well-founded ordering $\succ_R$ that will be used in the proof of Theorem 27. In contrast to the Horn and equational cases, it does not coincide with $\to_{R_S}^+$.

DEFINITION 24. *Let S be a set of clauses. By $\succ_R$ we denote the smallest transitive relation such that $s \succ_R t$ whenever (i) $s \to_{R_S}^+ t$ or (ii) $s \rhd t$ or (iii) s and t are irreducible at non-topmost positions w.r.t. R_S and $s \succ t$.*

The ordering $\succ_R$ fulfils the following properties wrt. R_S.

PROPERTY 25. *Let S be a set of clauses. Then for all ground terms s and t s.t. s is irreducible at non-topmost positions and $s \succ_R t$ we have*

1. *$s \succ t$*
2. *t is irreducible at non-topmost positions*

Proof: If $s \succ_R t$ by case (ii) or (iii) it trivially holds. Otherwise, $s \to_{R_S} s_1 \to_{R_S} \dots \to_{R_S} s_n \to_{R_S} t$. Since s is irreducible at non-topmost positions and, by property 21.1, all right hand sides of rules in R_S are irreducible at non-topmost positions, all steps in the sequence are at topmost positions, which implies on one hand that t is irreducible at non-topmost positions, and on the other, by definition of R_S, that $s \succ s_1 \succ \dots \succ s_n \succ t$. $\square$

LEMMA 26. *Let S be a set of clauses. Then $\to_{R_S}^+$ is a reduction ordering and $\succ_R$ is well-founded.*

Proof: Since R_S is a terminating TRS, $\to_{R_S}^+$ is a reduction ordering by Lemma 6. Now assume $\succ_R$ is not well-founded. Since $(\to_{R_S}^+ \cup \rhd)^+$ is well-founded by Lemma 3, there is also some such infinite sequence $t_1 \succ_R t_2 \succ_R t_3 \succ_R \dots$ starting with case (iii): where t_1 and t_2 are irreducible w.r.t. R_S at non-topmost positions and $t_1 \succ t_2$. Then, by Property 25, t_3, t_4 etc, are all irreducible at non-topmost positions, and $t_1 \succ t_2 \succ t_3 \dots$ w.r.t. the well-founded west ordering $\succ$, which is a contradiction. $\square$

THEOREM 27. *(refutation completeness of $\mathcal{I}$ for general clauses) Let S be a set of clauses closed under $\mathcal{I}$ with respect to a west ordering $\succ$. Then $\square \in S$ if, and only if, S is unsatisfiable.*

Proof: We prove that if $\Box \notin S$ then R_S^* is a model of S by induction on $(\succ_R)_c$, which is a well-founded ordering on clauses. In the following, we (ambiguously) write $\succ_R$ for terms, equations and clauses instead of $\succ_R$, $(\succ_R)_e$ and $(\succ_R)_c$ respectively. We derive a contradiction from the existence of a minimal w.r.t. $\succ_R$ ground instance C of a clause in S such that $R_S^* \not\models C$.

1. We first consider the case where C is an instance with σ of a positive clause $\to s \simeq t, \Delta$, where $s\sigma \simeq t\sigma$ is strictly maximal with respect to $\succ_e$ in C and w.l.o.g. $s\sigma \succ t\sigma$. Since $R_S^* \not\models C$, we know C has not generated any rule due to one of the following reasons:

1a. $t\sigma$ is reducible by R_C at a non-topmost position. Then there exists some clause $\to l \simeq r, \Delta'$ in S whose instance C' with $C \succ_c C'$ generates the rule $l\sigma \to r\sigma$ reducing $t\sigma$.

If, for some prefix p' of p, the term $t|_{p'}$ is a variable x, then the same lifting argument as in proof of Theorem 11 applies.

Otherwise, $t|_p$ is a non-variable subterm of t, and there exists an inference by paramodulation right:

$$\frac{\to l \simeq r, \Delta' \qquad \to s \simeq t, \Delta}{(\to s \simeq t[r]_p, \Delta, \Delta')\theta}$$

whose conclusion has an instance D of the form $(\to s \simeq t[r]_p, \Delta, \Delta')\sigma$ such that by Lemma 23 $R_S^* \not\models D$. This contradicts the minimality of C since $C \succ_R D$ for the following reasons: $mse(C) = mse(\Delta) \cup \{\ \{s\sigma, t\sigma\}\ \}$ and $mse(D) = mse(\Delta) \cup \{\ \{s\sigma, t\sigma[r\sigma]_p\}\ \} \cup mse(\Delta')$. Then we need to prove that $\{s\sigma, t\sigma\} \succ_{Rmul} \{s\sigma, t\sigma[r\sigma]_p\}$ and $(s\sigma \simeq t\sigma) \succ_R e\sigma$ for all equations e in Δ'. The first one holds since, by monotonicity of $\to_{R_S}^+$, we have $t\sigma[l\sigma]_p \succ_R t\sigma[r\sigma]_p$. For the second one, we have $l\sigma \succeq u\sigma$ for all terms u in Δ', and since they are also irreducible at non-topmost positions, we have $l\sigma \succeq_R u\sigma$. Furthermore $t\sigma \succ_R l\sigma$, since $t\sigma \rhd l\sigma$, which implies $t\sigma \succ_R u\sigma$ for all terms u in Δ', and hence we can conclude.

1b. $s\sigma$ is reducible by R_C and case 1a. does not apply. Then we argue as in the previous case by lifting or an inference in s whose conclusion has an instance D of the form $(\to s[r]_p \simeq t, \Delta, \Delta')\sigma$, contradicting the minimality of C. Here $C \succ_R D$ if $p \neq \lambda$ as in the previous case. If $p = \lambda$ then $(s\sigma \simeq t\sigma) \succ_R e\sigma$ for all equations e in Δ': since $C \succ_c C'$, and $s\sigma \simeq t\sigma$ is the strictly maximal equation of C, we have $(s\sigma \simeq t\sigma) \succeq_e (l\sigma \simeq r\sigma) \succ_e e\sigma$, and since all these equations are irreducible at non-topmost positions, $(s\sigma \simeq t\sigma) \succ_R e\sigma$.

1c. An equation $u\sigma \simeq v\sigma$ in Δ is reducible by R_C at a non-topmost position. The proof is like case 1a.

1d. None of the previous cases applies and Δ is of the form $s' \simeq t', \Delta'$ where $s\sigma \equiv s'\sigma$ and $R_S^* \models t\sigma \simeq t'\sigma$, that is, the last condition of Definition 20 fails. Then there exists some inference by equality factoring

$$\frac{\to s \simeq t, s' \simeq t', \Delta'}{(t \simeq t' \to s \simeq t, \Delta')\theta}$$

whose conclusion has an instance D such that $R_S^* \not\models D$. This contradicts the minimality of C: since $s\sigma$, $t\sigma$, and $t'\sigma$ are irreducible at non-topmost positions, and $s'\sigma \succ t\sigma$ and $s'\sigma \succ t'\sigma$, we have $C \succ_R D$.

2. If C is an instance with σ of $\to s \simeq t, \Delta$, where $s\sigma \simeq t\sigma$ is maximal but not strictly maximal with respect to $\succ_e$ in C, then condition 2 of Definition 20 fails. If $s\sigma \simeq t\sigma$ is reducible at non-topmost positions, then the same reasoning as in case 1a. applies. Otherwise, the proof of case 1d. applies.

3. If C is an instance with σ of a clause $\Gamma, \underline{s \simeq t} \to \Delta$, where $s\sigma \equiv t\sigma$, we conclude, as in the proof of Theorem 18, by equality resolution.

4. If C is an instance with σ of a clause $\Gamma, \underline{s \simeq t} \to \Delta$, where $s\sigma \not\equiv t\sigma$. Then $R_S^* \models s\sigma \simeq t\sigma$. Then, since R_S is convergent, there is a rewrite proof for $s\sigma \simeq t\sigma$. W.l.o.g. assume $s\sigma \succ t\sigma$. This implies that either $s\sigma$ is reducible or $t\sigma$ is reducible at a non-topmost position (otherwise the only possible reduction step is at the top of $t\sigma$ and a new term t' is obtained with $t\sigma \succ t'$ and t' again irreducible except at the top; since $s\sigma \succ t\sigma$, a sequence of topmost steps on $t\sigma$ can never produce $s\sigma$).

Then, as before, either the lifting argument applies or there is an inference by paramodulation left (we only develope the case where the step takes place in s):

$$\frac{\to l \simeq r, \Delta' \qquad \Gamma, \underline{s \simeq t} \to \Delta}{(\Gamma, s[r]_p \simeq t \to \Delta', \Delta)\theta}$$

whose conclusion has an instance D of the form $(\Gamma, s[r]_p \simeq t \to \Delta, \Delta')\sigma$ such that $R_S^* \not\models D$. This is a contradiction as before, since also here $C \succ_R D$ for the following reasons: we have $s\sigma \trianglerighteq l\sigma$ and hence $s\sigma \succeq_R l\sigma$. We also have $l\sigma \succ_R r\sigma$, and $l\sigma \simeq r\sigma \succ_R u\sigma \simeq v\sigma$ for all equations $u \simeq v$ in Δ' and hence $\{s\sigma, s\sigma, t\sigma, t\sigma\} \succ_{Rmul} \{l\sigma, r\sigma\} \succ_{Rmul} \{u\sigma, v\sigma\}$. $\square$

8. Unfailing Knuth-Bendix Completion

In this section, E denotes a set of equations and $\succ_r$ a reduction ordering on $T(\mathcal{F}, \mathcal{X})$. Then a *convergent TRS for E and $\succ_r$* is a convergent TRS, logically equivalent to E, and such that $l \succ_r r$ for all its rules $l \to r$. The problem we deal with is finding a convergent TRS for the given E and $\succ_r$ whenever it exists, and find it in finite time if it is finite.

8.1. A THEORETICAL PROCEDURE

It is not difficult to devise a procedure of theoretical nature, i.e., without much practical value, for finding a convergent TRS for E and $\succ_r$. The idea is to systematically enumerate all equational consequences of E, say $s_1 \simeq t_1, s_2 \simeq t_2, \ldots$ If a finite convergent TRS R exists, there exists some (probably huge) i, such that R is contained in (the orientations of) a subset of $\{s_1 \simeq t_1, \ldots, s_i \simeq t_i\}$. One can find this R by periodically checking during the enumeration process whether (i) the subset R of orientable (with $\succ_r$) rules of $\{s_1 \simeq t_1, \ldots s_i \simeq t_i\}$ is confluent and (ii) whether R entails E. The confluence of R can be decided by checking joinability of all its critical pairs. After this, entailment of E can be decided by rewriting.

The following lemma states that, after enumerating i equational consequences of E, it is not necessary to check for confluence of the orientations for each subset of $\{s_1 \simeq t_1, \ldots, s_i \simeq t_i\}$, but that it suffices to consider the orientations of the whole set $\{s_1 \simeq t_1, \ldots, s_i \simeq t_i\}$, because unnecessary rules do not destroy convergence.

LEMMA 28. *Let E be a set of equations, let $\succ_r$ be a reduction ordering on $T(\mathcal{F}, \mathcal{X})$, and let R be a convergent TRS for E and $\succ_r$. Let R' be any set of rules $l \to r$ such that $l \succ_r r$ and $E \models l \simeq r$. Then $R \cup R'$ is a convergent TRS for E and $\succ_r$.*

Proof: Clearly all critical pairs between rules in $R \cup R'$ are logical consequences of E. Since R is a convergent TRS for E, all these critical pairs have rewrite proofs by R, i.e., they are joinable. Hence $R \cup R'$ is a convergent TRS for E and $\succ_r$. □

The TRS R found in this way may not be minimal, that is, it may have some proper subset R' that is also a convergent TRS for E and $\succ_r$, but such a minimal TRS always exists, and it can be effectively computed from a finite R:

LEMMA 29 ((Dershowitz et al., 1988)). *For every convergent TRS its unique* canonical[1] *version can be obtained by* interreducing *it by (i) normalizing all right hand sides and (ii) removing all rules whose left hand sides are reducible by other rules.*

[1] Unfortunately, the word canonical is sometimes also used as equivalent of convergent.

8.2. Practical procedures

Regarding practically useful procedures, Devie showed that for left- and right *linear* E (i.e., no variable occurs more than once in a side of an equation) standard Knuth-Bendix completion finds R (Devie, 1990). For the general problem, the previously existing procedures still relied on the enumeration of all equational consequences (see e.g., (Devie, 1992)).

In the following, R will denote the canonical TRS for E and $\succ_r$ (we assume R exists), and we denote by R_g the canonical TRS for $gnd(R)$, the set of all ground instances of rules of R.

The following lemma strengthens the uniqueness result of (Dershowitz et al., 1988) to TRS included in a west ordering, but only for the ground case:

LEMMA 30. *Let $\succ$ be a west ordering, and let R_1 and R_2 be interreduced TRS over $T(\mathcal{F})$, both included in $\succ$, and such that $R_1^* = R_2^*$. Then $R_1 = R_2$.*

Proof: Consider the rule $l \rightarrow r$ in $(R_1 \cup R_2) \setminus (R_1 \cap R_2)$ with minimal l w.r.t. $\succ$. Assume w.l.o.g. $l \rightarrow r \in R_1$. Since $R_1^* \models l \simeq r$ and hence $R_2^* \models l \simeq r$ and R_2 is convergent, there must be a rewrite proof by R_2 for $l \simeq r$. But, since R_1 is interreduced, r and all strict subterms of l are irreducible w.r.t. R_1, and hence also w.r.t. R_2, since any rule reducing them has a lhs smaller than l w.r.t. $\succ$. Hence there is a rule $l \rightarrow r'$ in R_2. But then r' and r are both irreducible by R_2 and $R_2 \models r \simeq r'$. This implies that r and r' are the same term, contradicting the assumption that $l \rightarrow r \notin R_2$. $\square$

In the following we consider the ordered equational paramodulation rule given in Definition 7 w.r.t. a west ordering $\succ$ extending $\succ_r$, and the ground TRS R_E as in Definition 8. Then, by Property 9, for all rules $l \rightarrow r$ in R_E we have that $l \succ r$, that r is irreducible by R_E and that l is irreducible by $R_E \setminus \{l \rightarrow r\}$.

LEMMA 31. *Let E' be the closure of E under ordered equational paramodulation w.r.t. a west ordering $\succ$ extending $\succ_r$. Then $R_g = R_{E'}$.*

Proof: By Theorem 11 we have $R_{E'}^* \models E'$, and hence $R_{E'}^* \models E$, and hence $E^* = R_{E'}^*$ since all rules in $R_{E'}$ are logical consequences of E. Furthermore, we clearly have $E^* = R^* = R_g^*$, and hence $R_g^* = R_{E'}^*$. We also have $R_g \subseteq \succ_r \subseteq \succ$, and $R_{E'} \subseteq \succ$. Therefore, since both are interreduced we conclude by Lemma 30 that $R_{E'} = R_g$. $\square$

Now we come to the main theorem of this section. It says that R is a subset of the closure of E under the inference rule of ordered paramodulation (which we recall here from Definition 7):

$$\frac{l \simeq r \qquad s \simeq t}{(s[r]_p \simeq t)\sigma}$$

where $\sigma = mgu(s|_p, l)$ and $s|_p$ is not a variable, and with ordering restrictions saying that the inference is not needed if $r\sigma \succeq_r l\sigma$ (or, more generally, that the inference is needed only if $l\sigma\theta \succ r\sigma\theta$ for some ground substitution θ, where $\succ$ is the west ordering extending $\succ_r$).

THEOREM 32. *Let E' be the closure of E under equational ordered paramodulation w.r.t. a west ordering $\succ$ extending $\succ_r$. Then $E' \supseteq R$.*

Proof: W.l.o.g. assume there are sufficiently new constants in $\mathcal{F}$ that do not occur in E or R. Let $l \to r$ be an arbitrary rule in R. We prove that $l \simeq r \in E'$. Let σ be the ground substitution replacing each variable with a distinct new constant. Then $l\sigma \to r\sigma$ is in $gnd(R)$ and also in R_g: since R is interreduced, r is irreducible w.r.t. R (and hence $r\sigma$ by $gnd(R)$), and l is irreducible w.r.t. $R \setminus \{l \to r\}$ (and hence $l\sigma$ by $gnd(R) \setminus \{l\sigma \to r\sigma\}$). If $l\sigma \to r\sigma$ is in R_g and by the previous lemma $R_g = R_{E'}$, then $l\sigma \to r\sigma$ is in $R_{E'}$. Since the new constants do not occur in E, if $l\sigma \to r\sigma \in R_{E'}$, then some $l' \simeq r'$ is in E', and $l\sigma \equiv l'\theta\sigma$ and $r\sigma \equiv r'\theta\sigma$ for some θ. We conclude by showing that θ is the identity substitution: there is no equation $l' \simeq r'$ that strictly subsumes a rule in R; otherwise, the rewrite proof by R of $l' \simeq r'$ would apply some rule different from $l \to r$, which would then also reduce $l \to r$, contradicting $l \to r \in R$. □

9. Redundancy and constraints

In this section we shortly mention the applicability of techniques for *redundancy elimination* and *constraint inheritance* in the context of this article. We refer to (Bachmair and Ganzinger, 1994; Bachmair et al., 1995; Bachmair and Ganzinger, 1998; Nieuwenhuis and Rubio, 2001) for all details, for which it would make no sense to repeat them here.

There are standard ways for uniformly covering simplification and deletion techniques that are compatible with refutation completeness by notions of redundancy for inferences and clauses, where *saturation* amounts to the closure under $\mathcal{I}$ *up to redundant inferences.*

In the setting of this article, there is an important difference, however, because two different orderings are considered. The inferences are computed w.r.t. the west ordering $\succ$, and the completeness proof uses the reduction ordering $\succ_R$. Unfortunately, redundancy should hence be defined w.r.t. $\succ_R$, which is unknown during the saturation process. But in many cases it is clear that $\succ_R$ can be sufficiently approximated. For example, practical redundancy notions like tautology deletion or subsumption are (trivially) correct w.r.t. any $\succ_R$.

Ordered paramodulation is a very adequate inference rule for dealing with constrained clauses and the *basic* strategy. These ideas are directly applicable here: subterms created by unifiers of inferences on ancestors can be *blocked* for inferences, and in our setting this is true also for proper subterms of the term $r\sigma$ (which are irreducible by R_S in our proof) in the conclusions of paramodulation inferences.

10. Future work

After the results of this article several open problems become of interest. In this section some of them are described.

1. Now it becomes interesting to explore completeness results for deduction modulo equational theories E for which monotonic E-compatible total orderings do not exist, but that can be handled with the techniques described in this article.

2. It has to be more carefully explored to what extent other redundancy notions, like demodulation, are applicable in our framework for ordered paramodulation.

3. In this article we have dealt with inference systems with eager selection of negative literals, that is, a negative literal is selected whenever there is any. For the moment we know of no *direct* way of proving the completeness for strategies with selection of (maximal) positive literals. However, in (Bofill and Godoy, 2001) a relatively simple proof transformation technique is given for the Horn case, by which the completeness is shown for arbitrary selection strategies and arbitrary paramodulation-based inference systems that are complete with constraint inheritance and eager selection of negative literals. However, there exist examples showing the incompatibility of these techniques with tautology deletion (see Example 36 below). This makes it unlikely that a standard model construction-based proof, where such tautologies are redundant, can be found.

Regarding Knuth-Bendix completion, after the results of this article the following interesting questions remain unanswered:

4. Does Theorem 32 still hold when simplification by rewriting with respect to the reduction ordering $\succ_r$ is applied?

5. Does it hold when the strict ordered paramodulation rule of Definition 12 (where inferences at topmost positions of small sides w.r.t. $\succ$ are not needed) is used?

6. Would it hold when inferences with (or on) small sides w.r.t. $\succ_r$ are not computed at all? Or, in other words, is *superposition* w.r.t. arbitrary reduction orderings complete?

10.1. Counterexamples

We are also working on more restrictive inference systems. But a number of negative results have been obtained, which are best described by the following counterexamples.

EXAMPLE 33. *Inferences on non-maximal positive atoms are needed: if the inconsistent set S consists of the clauses*

$$\begin{array}{rl} & \rightarrow P(b,b), P(a,b) \\ \underline{P(b,b)}, P(a,b) & \rightarrow \\ & \rightarrow a \simeq b \end{array}$$

with a west ordering $\succ$ where $a \succ b$ and $P(b,b) \succ P(a,b)$, then the only other possible inference produces the tautology $P(a,b) \rightarrow P(a,b)$.

EXAMPLE 34. *Also inferences on small sides of positive equations are needed: if the inconsistent set S consists of the clauses*

$$\begin{array}{rl} & \rightarrow a \simeq b \\ & \rightarrow g(b) \simeq g(g(a)) \\ x \simeq g(x) & \rightarrow \end{array}$$

with a west ordering $\succ$ where $a \succ b$ and $g(b) \succ g(g(a))$, then the only other possible inferences produce clauses of the form $b \simeq g^n(b) \rightarrow$ and $b \simeq g^n(a) \rightarrow$ for $n > 1$.

EXAMPLE 35. *Also inferences on small sides of negative equations are needed: if the inconsistent set of clauses S is*

$$\begin{array}{rl} & \rightarrow a \simeq b \\ f(x,x) \simeq f(a,b) & \rightarrow \end{array}$$

with a west ordering $\succ$ where $f(a,a) \succ f(a,b)$ and $f(b,b) \succ f(a,b)$, then S is closed.

EXAMPLE 36. *(*Lynch, 1997*) For arbitrary selection strategies, these techniques are incompatible with tautology deletion. Consider the set*

$$\begin{array}{lll} 1. & & \rightarrow P(c,b,b) \\ 2. & P(c,c,b), P(c,b,c) & \rightarrow b \simeq c \\ 3. & P(x,y,y) & \rightarrow P(x,y,x) \\ 4. & P(x,y,y) & \rightarrow P(x,x,y) \\ 5. & P(c,c,c) & \rightarrow \end{array}$$

This clause set is inconsistent: from 1. and 3. we get $P(c,b,c)$, and from 1. and 4. we get $P(c,c,b)$; these two atoms together with 2. produce $b \simeq c$, which gives the empty clause with 1. and 5. But the empty clause cannot be obtained by ordered paramodulation on non-tautology clauses with an ordering where $b \succ c$, and where always the positive literals are selected (except in clause 5., which has none). In fact, the only new clauses obtained are tautologies.

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