Redundancy Notions for Paramodulation with Non-Monotonic Orderings

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Abstract. Recently, ordered paramodulation and Knuth-Bendix completion were shown to remain complete when using non-monotonic orderings. However, these results only implied the compatibility with too weak redundancy notions and, in particular, demodulation could not be applied at all.
In this paper, we present a complete ordered paramodulation calculus compatible with powerful redundancy notions including demodulation, which strictly improves previous results.
Our results can be applied as well to obtain a Knuth-Bendix completion procedure compatible with simplification techniques, which can be used for finding, whenever it exists, a convergent TRS for a given set of equations and a (possibly non-totalizable) reduction ordering.

1 Introduction

Knuth-Bendix-like completion techniques and their extensions to ordered paramodulation for first-order clauses are among the most successful methods for automated deduction with equality [BG98,NR01]. For many years, all known completeness results for Knuth-Bendix completion and ordered paramodulation required the term ordering $\succ$ to be well-founded, monotonic and total (or extendable to a total ordering) on ground terms [HR91,BDH86,BD94,BG94]. In [BGNR99,BGNR03], the monotonicity requirement was dropped and well-foundedness and the subterm property were shown to be sufficient for ensuring refutation completeness of ordered paramodulation (note that any such ordering can be totalized without losing these two properties). And recently, in [BR02], it was shown that well-foundedness of the ordering suffices for completeness of ordered paramodulation for Horn clauses, i.e., the subterm property can be dropped as well. Apart from its theoretical value, these results have several potential applications in contexts where the usual requirements are too strong.

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For example, in deduction modulo built-in equational theories E, where E-*compatibility* of the ordering (i.e. $s =_E s' \succ t' =_E t$ implies $s \succ t$) is needed, finding E-*compatible* orderings fulfilling the required properties is extremely complex or even impossible. For instance, when E contains an idempotency axiom $f(x, x) \simeq x$, no total E-compatible reduction ordering exists: if $s \succ t$, by monotonicity one should have $f(s, s) \succ f(s, t)$, which by E-compatibility implies $s \succ f(s, t)$, and hence non-well-foundedness. Therefore, the techniques for dropping ordering requirements, among other applications, open the door to deduction modulo many more classes of equational theories.

However, ordered strategies are useful in practice only if compatibility with redundancy is shown. Simplification of formulae and elimination of redundant formulae are essential components of automated theorem provers. In fact, in most successful automated theorem provers, simplification is the primary mode of computation, whereas prolific deduction rules are used only sparingly.

Here, we present a calculus which strictly improves the one in [BGNR03]. On the one hand, regarding the amount of inferences needed to be performed, our inference system is essentially the same as the one in [BGNR03], but, on the other hand, our calculus is compatible with powerful redundancy notions which include simplification by rewriting (i.e. demodulation). As a side effect, as in [BGNR03], we can apply our results to obtain a Knuth-Bendix completion procedure, but in this case compatible with simplification techniques. This procedure can be used for finding, whenever it exists, a convergent TRS for a given set of equations and a (possibly non-totalizable) reduction ordering.

In our calculus it is assumed that there is a known reduction ordering $\succ_r$, which is included in the (possibly non-monotonic) ordering $\succ$ w.r.t. which equations are oriented. Then we show that redundancy notions w.r.t. the reduction ordering $\succ_r$ can be applied while keeping refutation completeness. Roughly, a formula is redundant if it is a logical consequence of smaller (w.r.t. a well-founded ordering including $\succ_r$) formulae. Then demodulation w.r.t. $\succ_r$ fits in this abstract notion of redundancy since, in particular, a formula which has been simplified by rewriting follows from its smaller w.r.t. $\succ_r$ (due to monotonicity of $\succ_r$) "simplified version" and the (also smaller) equations used to simplify it. As a particular case, if all equations involved in the saturation process[3] turn out to be orientable with $\succ_r$, then demodulation can be fully applied. The following well-known simple challenge example illustrates this situation:

Example 1. Under our inference system,

$$f(f(x)) \simeq f(g(f(x)))$$

is saturated if $f(f(x)) \succ_r f(g(f(x)))$, and the ordering $\succ$ w.r.t. which ordered paramodulation is applied is an extension of $\succ_r$ (which cannot be totalized without losing monotonicity, since $g(f(x))$ and $f(x)$ should be incomparable in any monotonic extension). As shown below, for every possible inference, the

[3] The saturation of a set of clauses S amounts to the closure of S under the inference system *up to redundancy*.

conclusion can be simplified into a tautology. By an ordered paramodulation with

$$f(f(x)) \simeq f(g(f(x)))$$

on itself at the underlined position of

$$f(\underline{f(y)}) \simeq f(g(f(y))),$$

where the most general unifier of $f(f(x))$ and $f(y)$ is $\{y \mapsto f(x)\}$, we get

$$f(f(g(f(x)))) \simeq f(g(f(f(x)))).$$

Now, by previously renaming the variables, both sides of the conclusion

$$f(f(g(f(z)))) \simeq f(g(f(f(z))))$$

can be rewritten in one step with $f(f(x)) \simeq f(g(f(x)))$ (oriented with the reduction ordering) into $f(g(f(g(f(z)))))$, i.e., the conclusion of the inference can be simplified into a tautology. On the other hand, by an ordered paramodulation with

$$f(f(x)) \simeq f(g(f(x)))$$

on itself at the underlined position of

$$f(f(y)) \simeq f(g(\underline{f(y)})),$$

we get

$$f(f(f(x))) \simeq f(g(f(g(f(x))))),$$

i.e. $f(f(f(z))) \simeq f(g(f(g(f(z)))))$. In this case, $f(f(f(z)))$ can be rewritten in two steps with $f(f(x)) \simeq f(g(f(x)))$ (oriented with the reduction ordering) into $f(g(f(g(f(z)))))$, and hence the conclusion of the inference can be simplified into a tautology as before. Therefore, the system is saturated. It is worth noting that, without considering demodulation, the closure of $f(f(x)) \simeq f(g(f(x)))$ under ordered paramodulation w.r.t. $\succ$ diverges. $\square$

We must point out that, although demodulation w.r.t. $\succ_r$ is allowed in our inference system, not every clause can be demodulated: during our saturation process, some subterms of clauses become "blocked" for demodulation. Roughly, we block for demodulation those subterms that occur as the small side (w.r.t. $\succ$) of an equation that cannot be handled with $\succ_r$. Later inferences will inherit these blockings. Note that, if paramodulation is applied w.r.t. a non-monotonic ordering $\succ$ and we use full demodulation w.r.t. $\succ_r$, we can lose completeness as shown in Example 4 of Section 7.

The reason for adding these blockings comes from our completeness proof, which is based on the model generation technique of [BG94] and its variant used in [BGNR03]. In the latter, in contrast to the former, the ordering used for orienting the equations in the inference system and the ordering used for induction in the completeness proof do not coincide. Concretely, equations are oriented w.r.t. a non-monotonic ordering $\succ$, whereas completeness is proved by induction w.r.t. $\xrightarrow{+}_R$, where R is the limit ground rewrite system that defines the model. This is the key for the completeness proof of ordered paramodulation with non-monotonic orderings given in [BGNR03]. Since completeness is proved by induction w.r.t. $\xrightarrow{+}_R$, the redundancy notions should be defined w.r.t. $\xrightarrow{+}_R$ as well, instead of w.r.t. $\succ$, which is what would be desirable in principle. Unfortunately, $\xrightarrow{+}_R$ is unknown during the saturation process, and it is not clear how it can be approximated sufficiently. This is why in [BGNR03] it was left open to what extent demodulation could be applied (although some practical redundancy notions like tautology deletion and subsumption were shown to preserve completeness).

Here our aim is to add for instance demodulation w.r.t. the reduction ordering $\succ_r$, which is included in the (possibly)non-monotonic extension $\succ$. The idea for doing that is, roughly, to find a well-founded ordering $\succ_i$, which combines as much as possible $\succ_r$ with $\xrightarrow{+}_R$, and then prove completeness by induction w.r.t. $\succ_i$. The problem is that, although $\succ_r$ is a well-founded and monotonic ordering and R is a terminating TRS whose rules are included in a well-founded extension $\succ$ of $\succ_r$, the relation $\rightarrow_R \cup \succ_r$ is not well-founded in general and, in fact, $\xrightarrow{+}_R$ can even contradict $\succ_r$. Take e.g. $f(b) \succ_r f(a)$ and a well-founded extension $\succ$ of $\succ_r$ such that $f(b) \succ f(a) \succ a \succ b$. Then possibly $a \rightarrow b \in R$ (since the rules in R are oriented w.r.t. $\succ$) and hence the infinite sequence $f(a) \rightarrow_R f(b) \succ_r f(a) \cdots$ can be built.

The idea to circumvent this non-well-foundedness problem is to block the "bad" right-hand sides, i.e., the ones that can be introduced by a rewriting step with R which is not included in $\succ_r$. Then we can combine $\succ_r$ with $\rightarrow_R$ if the comparisons with $\succ_r$ cannot take into account the blocked terms. For instance, in the previous example, b should be blocked, and then $f(b)$ with b blocked is no longer greater than $f(a)$.

A convenient way to represent terms with blocked positions is by means of superindexed subterms, also called *marked terms*. For example $f(b^x)$, where x is a variable, denotes the term $f(b)$ where the subterm b is blocked. Then, although for performing inferences $f(b^x)$ still corresponds to the term $f(b)$, for the redundancy notions it will be seen as $f(x)$ with a blocked term b in x. In this context $f(b^x)$ cannot be simplified with $f(b) \rightarrow f(a)$, but it could still be simplified with $f(x) \rightarrow g(x)$ into $g(b^x)$, if $f(x) \succ_r g(x)$. Marked terms resemble the term closures of [BGLS95]. However, their semantics is fairly different: here the blockings have only effect on the redundancy notions, i.e., the ordered paramodulation inferences are applied at both blocked and unblocked positions.

Detection of terms that have to be marked and introduction of marks in the clauses will be part of our inference system. The terms to be marked come from the equations that cannot be oriented with $\succ_r$ (since we do not know in advance which equations will be part of R). These terms are kept in a set, which is used for introducing marks in the clauses. Hence, if there are many equations that cannot be handled by the reduction ordering then many terms will become blocked and we will have almost no redundancy, but the same amount of inferences by paramodulation as in [BGNR03]. On the other hand, if many equations can be handled by the reduction ordering, then few terms will be blocked and we will have as much redundancy as in the paramodulation calculi for total reduction orderings.

The paper is structured as follows. Preliminaries are presented in Sect. 2. Marked terms and orderings on marked terms are defined in Sect. 3. In Sect. 4, our calculus, including the notions of redundancy for inferences, is presented for deduction with sets of equations. It is extended to Horn clauses in Sect. 5. Saturation derivations including redundancy of clauses are considered in Sect. 6. In Sect. 7 some practical notions of redundancy for Horn clauses are presented. In Sect. 8 we show how, from our results, a Knuth-Bendix completion procedure for finding convergent TRSs can be obtained. In Sect. 9 we extend our results to the case of general first order clauses. Finally in Sect. 10 we give some conclusions.

2 Preliminaries

2.1 Terms, Equations and Clauses

We use the standard definitions of [NR01]. $T(\mathcal{F}, \mathcal{X})$ ($T(\mathcal{F})$) is the set of (ground) terms over a set of symbols $\mathcal{F}$ and a denumerable set of variables $\mathcal{X}$ (over $\mathcal{F}$). The subterm of t at *position* p is denoted by $t|_p$, the result of replacing $t|_p$ by s in t is denoted by $t[s]_p$, and syntactic equality of terms is denoted by $\equiv$. A *substitution* is a partial mapping from variables to terms. The application of a substitution σ to a term t is denoted by $t\sigma$. The composition of two substitutions σ_1 and σ_2, denoted by juxtaposition, is defined as the composition of two functions, that is, $t\sigma_1\sigma_2 = (t\sigma_1)\sigma_2$.

An *equation* is a multiset of terms $\{s, t\}$, denoted $s \simeq t$ or, equivalently, $t \simeq s$. A first-order clause is a pair of finite multisets of equations Γ (the *antecedent*) and Δ (the *succedent*), denoted by $\Gamma \to \Delta$. Equations in Γ are called *negative literals* and equations in Δ are called *positive literals*. A Horn clause is a clause with at most one positive literal. The empty clause $\Box$ is a clause where both Γ and Δ are empty.

2.2 Terms and (Rewrite) Relations

If $\to$ is a binary relation, then $\leftarrow$ is its inverse[4], $\leftrightarrow$ is its symmetric closure, $\xrightarrow{+}$ is its transitive closure and $\xrightarrow{*}$ is its reflexive-transitive closure. If $s \xrightarrow{*} t$ and there is no t' such that $t \to t'$ then t is called *irreducible* and a *normal form* of s (w.r.t. $\to$). A relation $\to$ is *well-founded* or *terminating* if there exists no infinite sequence $s_1 \to s_2 \to \cdots$. The composition of two relations is indicated by $\circ$. Thus $s \xleftarrow{*} \circ \xrightarrow{*} t$ if there is an element r such that $s \xleftarrow{*} r$ and $r \xrightarrow{*} t$. A relation $\to$ is *confluent* or *Church-Rosser* if the relation $\xleftarrow{*} \circ \xrightarrow{*}$ is contained in $\xrightarrow{*} \circ \xleftarrow{*}$. A relation $\to$ on terms is *monotonic* if $s \to t$ implies $u[s]_p \to u[t]_p$ for all terms

[4] Here, and everywhere, we use the mirror image of a binary relation symbol for its inverse.

s, t and u and positions p. An *equivalence relation* is a reflexive, symmetric and transitive binary relation. A *congruence* is a monotonic equivalence relation.

A *rewrite rule* is an ordered pair of terms (s, t), written $s \to t$, and a set of rewrite rules R is a *term rewrite system* (TRS). The *rewrite relation* with R on $T(\mathcal{F}, \mathcal{X})$, denoted by $\to_R$, is the smallest monotonic relation such that for all rules $l \to r$ in R, and substitutions σ, $l\sigma \to_R r\sigma$. If $s \to_R t$ then we say that *s rewrites into t* with R. R is called terminating, confluent, etc. if $\to_R$ is. It is called *convergent* if it is confluent and terminating. The congruence $\overset{*}{\leftrightarrow}_R$ defines an *equality* Herbrand interpretation where $\simeq$ is interpreted by $s \simeq t$ iff $s \overset{*}{\leftrightarrow}_R t$. Such an interpretation will be denoted by R^*.

A (strict partial) *ordering* $\succ$ on $T(\mathcal{F}, \mathcal{X})$ is an irreflexive and transitive binary relation. It is *stable under substitutions* if $s \succ t$ implies $s\sigma \succ t\sigma$ for all substitutions σ. Monotonic orderings that are stable under substitutions are called *rewrite orderings*. A *reduction ordering* is a well-founded rewrite ordering.

A rewrite system terminates if, and only if, its rules are contained in a reduction ordering. In fact, if R is a terminating TRS, then $\overset{+}{\to}_R$ is a reduction ordering. However in restricted cases checking the rules with a well-founded ordering is enough:

Lemma 1. *[BGNR03] Let $\succ$ be a well-founded ordering, and let R be a ground TRS such that, for all $l \to r$ in R, $l \succ r$ and r is irreducible by R at non-topmost positions. Then R is terminating.*

A *quasi-ordering* $\succeq$ is a reflexive and transitive binary relation; the associated equivalence relation $\sim$ is the intersection of $\succeq$ with its inverse; the associated strict partial ordering $\succ$ is their difference. Hence, $\succeq$ is the disjoint union of $\succ$ and $\sim$. We say that a relation $\succ$ is compatible with an equivalence relation $\sim$ if $s \sim s' \succ t' \sim t$ implies $s \succ t$.

Proposition 1. *If $\succ$ is an ordering which is compatible with an equivalence relation $\sim$, then $\succ \cup \sim$ is a quasi-ordering whose associated strict partial ordering is $\succ$.*

Let $\sim$ be an equivalence relation and $\succ$ an ordering. The *multiset extension of* $\sim$ is defined as the smallest relation $\sim^{mul}$ on multisets of elements such that

$$\emptyset \sim^{mul} \emptyset$$

and

$$S \cup \{s\} \sim^{mul} S' \cup \{t\} \quad \text{if} \quad S \sim^{mul} S' \text{ and } s \sim t.$$

The *multiset extension of $\succ$ with respect to $\sim$* is defined as the smallest ordering $\succ^{mul}$ on multisets of elements such that

$$M \cup \{s\} \succ^{mul} N \cup \{t_1, \ldots, t_n\} \quad \text{if} \quad M \sim^{mul} N \text{ and } s \succ t_i \text{ for all } i \in 1 \ldots n.$$

Sometimes the notation $\succ^{mul}$ is used without explicitly indicating which is the equivalence relation $\sim$. In these cases $\sim$ is assumed to be the syntactic equality relation. If $\succ$ is well-founded on a set S, so is $\succ^{mul}$ on finite multisets over S.

There are some (quasi-)orderings that play a central role in our results (in particular, in the redundancy notions). The *subsumption* relation, denoted by $\geq$, is a quasi-ordering defined as $s \geq t$ if s is an instance of t, i.e., $s \equiv t\sigma$ for some σ. The equivalence relation associated with $\geq$ is denoted by $\doteq$. Note that if $s \doteq t$ then t is a variable renamed version (or a *variant*) of s. The *encompassment* relation, denoted by $\trianglerighteq$, is a quasi-ordering defined as $s \trianglerighteq t$ if a subterm of s is an instance of t, i.e., $s \trianglerighteq t\sigma$ for some σ; therefore, encompassment is the composition of the subterm and the subsumption relations. It is folk knowledge that, if $\succ_r$ is a reduction ordering, then the relation $\triangleright \cup \succ_r$ is well-founded.

It is said that an ordering fulfills the *subterm property* if $\succ \supseteq \triangleright$, where $\triangleright$ denotes the strict subterm relation. A *west ordering* [BGNR03] is a well-founded ordering on $T(\mathcal{F})$ that fulfills the subterm property and that is total on $T(\mathcal{F})$ (it is called *west* after well-founded, subterm and total). Every well-founded ordering can be totalized [Wec91], and hence every well-founded ordering satisfying the subterm property can be extended to a west ordering. We also have that every reduction ordering can be extended to a west ordering [BGNR03].

3 Marked Terms

Marked terms resemble the closures of [BGLS95]. A *term closure*, following the definition of [BGLS95], is a pair $s \cdot \gamma$ consisting of a term s (the *skeleton*) and a substitution γ from variables to terms. For example, $f(x, g(y)) \cdot \{x \mapsto a, y \mapsto h(z)\}$ is a closure with skeleton $f(x, g(y))$ and substitution $\{x \mapsto a, y \mapsto h(z)\}$. We generalize closures in that the substitution can map a variable not only to a term but also to a closure. For example, $f(x, g(y)) \cdot \{x \mapsto a, y \mapsto h(z) \cdot \{z \mapsto b\}\}$ is a generalized term closure, called a *marked term*. Moreover, closures are used in [BGLS95] to express that certain positions are blocked for inferences, whereas in our case they are blocked only for redundancy purposes. In order to ease the reading, sometimes we will denote marked terms by superindexing the "marked" subterms with variables. For example,

$$f(x, g(y)) \cdot \{x \mapsto a, y \mapsto h(z) \cdot \{z \mapsto b\}\}$$

will be written as

$$f(a^x, g(h(b^z)^y)).$$

Marked terms are extended in the natural way to equations and clauses. An equation between marked terms is called a *marked equation*, and a clause on marked equations is called a *marked clause*. Given a substitution γ of the form $\{x_1 \mapsto t_1 \cdot \gamma_1, \ldots, x_n \mapsto t_n \cdot \gamma_n\}$, the domain of γ, denoted by $Dom(\gamma)$, is defined as the set of variables $\{x_1, \ldots, x_n\}$, and the range of γ, denoted by $Ran(\gamma)$, is defined as the set of marked terms $\{t_1 \cdot \gamma_1, \ldots, t_n \cdot \gamma_n\}$. The variables occurring in $Dom(\gamma)$ are called *marking variables*. We will restrict marked terms to those which fulfill that marking variables occur at most once in the skeleton. For example $f(x, y) \cdot \{x \mapsto a, y \mapsto a\}$ is a valid marked term, but $f(z, z) \cdot \{z \mapsto a\}$

is not. Therefore, although we generalize closures in one sense, in another sense we restrict them.[5]

The congruence $\equiv$ on marked terms is defined as the monotonic, reflexive, symmetric and transitive closure including (i) $t^{xy} \equiv t^x$, and (ii) $s[t^x]_p \equiv s[t^y]_p$ if neither x nor y occur in $s[t]_p$.[6] Note that if $s \cdot \gamma \equiv t \cdot \delta$, then $s \doteq t$, i.e., s is a variant of t.

Let $s \cdot \{x_1 \mapsto t_1 \cdot \gamma_1, \ldots, x_n \mapsto t_n \cdot \gamma_n\}$ be a marked term and σ be a substitution of the form $\{y_1 \mapsto u_1 \cdot \delta_1, \ldots, y_m \mapsto u_m \cdot \delta_m\}$, such that the variables in the domains of all substitutions are distinct. Then $(s \cdot \gamma)\sigma$ is defined as

$$s\{y_1 \mapsto u_1, \ldots, y_m \mapsto u_m\} \cdot (\{x_1 \mapsto (t_1 \cdot \gamma_1)\sigma, \ldots, x_n \mapsto (t_n \cdot \gamma_n)\sigma\} \cup \delta_1 \cup \cdots \cup \delta_m),$$

where the marking variables of the resulting skeleton are renamed if necessary to keep linearity. For example,

$$f(x, g(x)^z)\{x \mapsto a^y\} = f(a^y, g(a^y)^z),$$

and

$$f(x, x)\{x \mapsto a^y\} = f(a^y, a^z).$$

By *Forget*$(s \cdot \gamma)$ we denote the term in $T(\mathcal{F}, \mathcal{X})$ obtained from $s \cdot \gamma$ by repeatedly applying all the substitutions. For example,

$$\mathit{Forget}(f(a^y, g(a^y)^z)) = f(a, g(a)).$$

A marked term $s \cdot \gamma$ is said to be ground if *Forget*$(s \cdot \gamma)$ is a ground term. If $(s \cdot \gamma)\sigma$ is a ground marked term, then σ is called a *ground substitution* for $s \cdot \gamma$, and $(s \cdot \gamma)\sigma$ is called a *ground instance* of $s \cdot \gamma$. This notation is extended to sets of terms, equations and clauses in the usual way.

The usual concepts of (and notation for) position, subterm and subterm replacement can be easily extended for marked terms. For example,

$$f(x, g(x)^z)|_2 = g(x)^z,$$

and

$$f(x, g(x)^z)[a^y]_{2 \cdot 1} = f(x, g(a^y)^z).$$

We write $u \trianglerighteq t \cdot \delta$ if $t \cdot \delta$ occurs in u, i.e., $u|_p \equiv t \cdot \delta$ for some position p, and we write $u \triangleright t \cdot \delta$ if $p \neq \lambda$. From these definitions, the notion of rewriting for terms can be extended to marked terms in the natural way.

Recall that, given two terms s and s' in $T(\mathcal{F}, \mathcal{X})$, we write $s \doteq s'$ if s' is a variant (that is, a variable renamed version) of s. This extends to marked terms

[5] Both the generalization and the restriction are necessary for our orderings on marked terms being well-founded.

[6] There is no condition that prevents terms like t^{xy} from appearing. For example, a marked variable z in e.g. z^y may be instantiated with a marked term t^x, giving t^{xy}. However, we will assume that these variable-variable bindings are instantiated whenever they arise, i.e. that terms are kept in its simpler form (t^x in the example).

by writing $s \cdot \gamma \doteq s' \cdot \gamma'$ if there are some substitutions σ and σ' such that $(s \cdot \gamma)\sigma \equiv s' \cdot \gamma'$ and $s \cdot \gamma \equiv (s' \cdot \gamma')\sigma'$. For example, if x, y, z and z' are variables and a is not a variable, then $h(g(z)^x) \doteq h(g(z')^y)$, whereas $h(g(a^z)^x) \not\doteq h(g(a)^y)$.

The concept of unifier between terms is extended to marked terms as follows: a unifier of two marked terms $s \cdot \gamma$ and $t \cdot \delta$ is a substitution σ such that $(s \cdot \gamma)\sigma \equiv (t \cdot \delta)\sigma$. There exists a unique (up to renaming of variables) most general unifier between marked terms, which can be obtained by straightforwardly adapting any known unification algorithm for terms.

3.1 Orderings on Marked Terms

Let $\succ_r$ be a reduction ordering, $\succ$ be a west ordering including $\succ_r$, and R be a terminating ground TRS. In this section we define the two main orderings on marked terms, namely $\succ_{red}$, which will be used in the redundancy notions and will be closely related to $\succ_r$, and $\succ_i$, which will be used for induction in the completeness proof and will combine as much as possible $\succ_{red}$ with $\xrightarrow{+}_R$. For the reasons explained in the introduction, it is crucial for the completeness proof that $\succ_{red} \subseteq \succ_i$.

We first define the marking ordering $\succ_m$. It only applies to terms that just differ in the markings and it is similar to the subsumption ordering for terms, in the sense that a term is smaller if it has higher marks.

Definition 1. *Let $s \cdot \gamma$ and $t \cdot \delta$ be two marked terms. Then $s \cdot \gamma \succ_m t \cdot \delta$ iff Forget$(s \cdot \gamma) \equiv$ Forget$(t \cdot \delta)$, $s \cdot \gamma \not\equiv t \cdot \delta$ and (i) $s \gtrdot t$ or (ii) $s \doteq t$ and Ran$(\gamma) \succ_m^{mul}$ Ran(δ).*

Given a marked term $s \cdot \gamma$, if $(s \cdot \gamma)|_p$ is of the form $(u \cdot \rho)^x$ then p is called a *marked position* of $s \cdot \gamma$. For example, 1 and $1 \cdot 1$ are marked positions of $g(f(a^x, b)^y)$. Observe that, if *Forget*$(s \cdot \gamma) \equiv$ *Forget*$(t \cdot \delta)$ and $s \cdot \gamma \not\equiv t \cdot \delta$, then $s \cdot \gamma$ and $t \cdot \gamma$ differ only in the marked positions. Then, if $s \cdot \gamma \succ_m t \cdot \delta$ by case (i), the skeleton t is the same as s except that some additional marking variables have been added. However, the number of marked positions of the whole marked term do not necessarily increase. For example, by case (i) we have

$$g(f(a, b)) \succ_m g(f(a^x, b)) \succ_m g(f(a^x, b^y)) \succ_m g(f(a, b))^z.$$

And by case (ii) we have, for example, $g(f(a, b))^z \succ_m g(f(a^x, b))^z$. Note also that, although $a \succ_m a^x$ by case (i), we do not have $a^x \succ_m a^{xy}$ by case (ii), since $a^x \equiv a^{xy}$.

Lemma 2. *$\succ_m$ is a well-founded ordering.*

Proof. It suffices to show that $\succ_m$ is a well-founded transitive relation. Given a marked term $s \cdot \gamma$, we define the interpretation $I(s \cdot \gamma)$ as the multiset of terms *Forget*$((s \cdot \gamma)|_p)$ for all non-marked positions p of $s \cdot \gamma$. It is not difficult to see that, if $s \cdot \gamma \succ_m t \cdot \delta$, then $I(s \cdot \gamma) \rhd^{mul} I(t \cdot \delta)$. Therefore well-foundedness of $\succ_m$ follows from well-foundedness of $\rhd^{mul}$. Transitivity of $\succ_m$ follows from transitivity of $\gtrdot$ and $\doteq$, and compatibility of $\gtrdot$ with $\doteq$. $\square$

Since $\succ_{red}$ will be used in the redundancy notions, all its ingredients are known while performing paramodulation steps. Given two marked terms, $\succ_{red}$ roughly compares the skeletons w.r.t. the reduction ordering and the encompassment relation and, if the skeletons are the same and the terms only differ in the markings, then it compares the level of the marks in the substitution part.

Definition 2. *Let $s \cdot \gamma$ and $t \cdot \delta$ be two marked terms. Then $s \cdot \gamma \succ_{red} t \cdot \delta$ iff (i) $s \rhd \cup \succ_r t$ or (ii) $s \doteq t$ and $Ran(\gamma) \succ_m^{mul} Ran(\delta)$.*

Since $\succ_i$ will be used for induction in the completeness proof, it extends $\succ_{red}$ with an additional case which uses as ingredient a terminating rewrite system R that will be known only in the proof.

Definition 3. *Let $s \cdot \gamma$ and $t \cdot \delta$ be two marked terms. Then $s \cdot \gamma \succ_i t \cdot \delta$ iff (i) $s \rhd \cup \succ_r t$ or (ii) $s \doteq t$ and $Ran(\gamma) \succ_m^{mul} Ran(\delta)$ or (iii) $s \doteq t$ and $Forget(Ran(\gamma)) \xrightarrow{+}_R Forget(Ran(\delta))$.*

In what follows, we will consider $\succ_i$ and $\succ_{red}$ as the transitive closures of the orderings defined above (composed with the congruence $\equiv$ on marked terms).

Lemma 3. *$\succ_i$ is well-founded.*

Proof. First of all observe that, in a sequence $s_1 \cdot \gamma_1 \succ_i s_2 \cdot \gamma_2 \cdots$, there can only be finitely many steps by case (i) since the relation $(\rhd \cup \succ_r) \circ \doteq$ is well-founded (see e.g. [DJ90]). Hence, from some point on, there are only steps by case (ii) or (iii). Now observe that, if $s_k \cdot \gamma_k \succ_i s_m \cdot \gamma_m \succ_i s_n \cdot \gamma_n$, where the first step is by case (ii) and the second step is by case (iii), then $s_k \cdot \gamma_k \succ_i s_n \cdot \gamma_n$ by case (iii), since $Forget(s_k \cdot \gamma_k) \equiv Forget(s_m \cdot \gamma_m)$ by definition of $\succ_m$. Then, since R is terminating, there can only be finitely many steps by case (iii), i.e., from some point on all steps must be by case (ii). Finally, since $\succ_m$ is well-founded by Lemma 2, there can only be finitely many steps by case (ii), which implies that $\succ_i$ is well-founded. □

Example 2. Here there are some examples of applications of $\succ_i$.

1. If $f(x) \succ_r g(x)$ for all x, then $f(a) \succ_i f(b^x) \succ_i g(b^x) \succ_i b^x$ by case (i) of the definition of $\succ_i$.
2. We have $h(g(f(a^x, a))^y) \succ_i h(g(f(a^x, a^z))^y) \succ_i h(g(f(a, a)^x)^y)$ by case (ii) of the definition of $\succ_i$.

4 Paramodulation with Equations

In this section we present an ordered paramodulation calculus for sets of equations. The calculus will work on both a set of marked equations E and a set of terms T, which contains the set of terms that are going to be marked.

In the following, let $\succ_r$ be a given reduction ordering and $\succ$ be a given west ordering including $\succ_r$. We assume that the non-marking variables of all equations are disjoint (if necessary, the variables can be renamed). On the other hand, to ease the reading, the marked terms of each equation are considered to have the same substitution (if necessary, the substitution can be extended and the marking variables of one of the two skeletons can be renamed).

Definition 4. *Let E be a set of marked equations and T be a set of terms, which satisfy the following condition: for every marked subterm $(s \cdot \gamma)^x$ occurring in E, we have that Forget$(s \cdot \gamma)$ is an instance of a term in T. The inference system $\mathcal{E}$ consists of the following two inference rules and a mark detection rule:*

Paramodulation:

$$\frac{l \cdot \delta \simeq r \cdot \delta \qquad s \cdot \gamma \simeq t \cdot \gamma}{(s \cdot \gamma[r \cdot \delta]_p \simeq t \cdot \gamma)\sigma} \qquad \textit{if}$$

1. *$\sigma = mgu(l \cdot \delta, s \cdot \gamma|_p)$, the most general unifier of $l \cdot \delta$ and $s \cdot \gamma|_p$,*
2. *Forget$(s \cdot \gamma|_p)$ is not a variable,*
3. *$l \succ_r r$ or $r \cdot \delta$ is marked on top, and*
4. *for some ground substitution θ, we have Forget$((l \cdot \delta)\sigma\theta) \succ$ Forget$((r \cdot \delta)\sigma\theta)$ and, if $p = \lambda$, then we also have Forget$((s \cdot \gamma)\sigma\theta) \succ$ Forget$((t \cdot \gamma)\sigma\theta)$.*

Marking:

$$\frac{s \cdot \gamma[u \cdot \delta]_p \simeq t \cdot \gamma}{(s \cdot \gamma[(u \cdot \delta)^x]_p \simeq t \cdot \gamma)\sigma} \qquad \textit{if}$$

1. *$\sigma = mgu(\textit{Forget}(u \cdot \delta), u')$ for some term u' in T,*
2. *u is not a variable, or u' is a variable and $p = \lambda$, and*
3. *x is a fresh variable.*

Mark detection:

$$T \Longrightarrow T \cup \{\textit{Forget}(r \cdot \delta)\} \qquad \textit{if}$$

1. *$l \cdot \delta \simeq r \cdot \delta \in E$,*
2. *$l \not\succ_r r$ and $r \cdot \delta$ is not marked on top, and*
3. *for some ground substitution θ, we have that Forget$((l \cdot \delta)\theta) \succ$ Forget$((r \cdot \delta)\theta)$.*

Some examples of applications of these rules can be found in Example 4 of Sect. 7.

4.1 Redundancy Notions: a Static View

Here we define some abstract redundancy notions for inferences, like the ones in [BG94] and [NR01]. Some practical redundancy notions fitting in these abstract notions are discussed afterwards, in Sect. 7.

Given a multiset of marked terms S, by *Marks*(S) we denote the multiset of marked subterms of the form $(s \cdot \gamma)^x$ that occur in some marked term of S. Given a set of terms M and a substitution θ, we say that θ is a substitution *with* *marks in* M if every term t in $Forget(Marks(Ran(\theta)))$ is an instance of a term in M.

By $gnd(E)$ we denote the set of all ground instances of equations in E (where a substitution is a mapping from variables to marked terms). Given a set of terms M, by $gnd_M(E)$ we denote the set of ground instances $e\theta$ of equations e in E such that θ is a substitution with marks in M.

Let $>$ be a well-founded ordering on marked equations. By $E^{<e}$ we denote the set of all d in E such that $e > d$. An analogous notation is used for quasi-orderings $\geq$. We define $\succeq_{red}$ as $\succ_{red} \cup \equiv$, where $\equiv$ denotes the equivalence of marked terms as defined in Sect. 3. Note that, since $\succ_{red}$ is compatible with $\equiv$, $\succeq_{red}$ is a quasi ordering whose associated strict partial ordering is $\succ_{red}$ by Proposition 1. Now we define redundancy of inferences and mark detection rules. Some explanations are given below.

Definition 5. *Let* π *be an inference by* $\mathcal{E}$ *with premises* $e_1, \ldots e_n$ *and conclusion* d*. Then a* ground instance $\pi\theta$ of the inference π *is an inference by* $\mathcal{E}$ *with premises* $e_1\theta, \ldots e_n\theta$ *and conclusion* $d\theta$ *for some ground substitution* θ*.*

Let G *be a set of ground marked equations. A ground inference by* $\mathcal{E}$ *with premises* $e_1, \ldots e_n$ *and conclusion* d *is* redundant in G *if*

$$Forget(G^{\prec_{red}^{mul} e_n}) \cup Forget(G^{\preceq_{red}^{mul} d}) \models Forget(d).$$

Let E *be a set of marked equations. An inference by* $\mathcal{E}$ *with premises* $e_1, \ldots e_n$ *and conclusion* d *is* redundant in E *if, for all sets of terms* M *and ground substitutions* θ *with marks in* M*,* $\pi\theta$ *is redundant in* $gnd_M(E)$*.*

Let T *be a set of terms. A mark detection rule* $T \Longrightarrow T \cup \{t\}$ *is redundant in* T *if* t *is an instance of a term in* T*.*

Note that the set of terms M is universally quantified in the previous definition of redundancy for inferences. This is due to the fact that the set of terms T to be marked will change during the computation, and it is crucial to have a notion of redundancy which is preserved along the computation. Moreover, the set $G^{\preceq_{red}^{mul} d}$ may not be included in $G^{\prec_{red}^{mul} e_n}$ since, in general, the conclusion of an inference does not need to be smaller w.r.t. $\succ_{red}^{mul}$ than the rightmost premise.

Redundant inferences and mark detection rules are unnecessary, and therefore we are interested in computing the closure of the set of equations with respect to $\mathcal{E}$ up to redundancy.

Definition 6. *Let* E *be a set of equations and* T *be a set of terms. The pair* $\langle E, T\rangle$ *is* saturated *with respect to* $\mathcal{E}$ *if every inference by* $\mathcal{E}$ *is redundant in* E *and every mark detection rule is redundant in* T*.*

4.2 Model Generation

Given a set of marked equations E and a set of terms T, here we show how to build an equational model of E defined by means of a set of rules coming from ground instances of E and T. Since the model is defined inductively, we need to treat E and T in a uniform way. To this end, a new constant symbol $\perp$ is introduced and the west ordering is extended with $\perp$ as minimal element.

Let T_{E} denote the set of equations $t \simeq \perp$ for all t in T, and T_{G} denote the set of mark free ground instances of equations in T_{E}. The model will be defined inductively on the equations in T_{G} and $gnd_T(E)$ with respect to $\succ_{mod}$, a total well-founded ordering including: $(s \simeq t) \succ_{mod} (u \simeq v)$ if

1. $\mathit{Forget}(s \simeq t) \succ^{mul} \mathit{Forget}(u \simeq v)$.
2. $\mathit{Forget}(s \simeq t) \equiv^{mul} \mathit{Forget}(u \simeq v)$ and $(s \simeq t) \succ_m^{mul} (u \simeq v)$, which means that $u \simeq v$ has marks at higher positions than $s \simeq t$.

Now we define a rewrite system R_E from $gnd_T(E)$, which requires to define at the same time a rewrite system R_T from T_{G}.

Definition 7. *The rewrite system R_{ET} is the union of R_E and R_T. Let e be a ground (marked) equation in $gnd_T(E) \cup T_{\mathrm{G}}$.*

1. *If e is an instance $(s \cdot \gamma)\theta \simeq (t \cdot \gamma)\theta$ in $gnd_T(E)$ of some equation $s \cdot \gamma \simeq t \cdot \gamma$ in E, then it generates the rule $(s \cdot \gamma)\theta \to (t \cdot \gamma)\theta$ in R_E if*
   (a) $s \succ_r t$ or $t \cdot \gamma$ is marked on top,
   (b) $\mathit{Forget}((s \cdot \gamma)\theta) \succ \mathit{Forget}((t \cdot \gamma)\theta)$,
   (c) $(s \cdot \gamma)\theta$ is irreducible by $R_E^e \cup R_T^e$, and
   (d) $(t \cdot \gamma)\theta$ is irreducible by $R_E^e \cup R_T^e$ at non-topmost positions.
2. *If e is an instance $t\theta \simeq \perp$ in T_{G} of an equation $t \simeq \perp$ in T_{E}, then it generates the rule $u \cdot \delta \to (u \cdot \delta)^x$ in R_T if $u \cdot \delta$ is the normal form of $t\theta$ w.r.t. R_T^e and $u \cdot \delta$ is irreducible by R_E^e,*

where R_E^e and R_T^e denote the set of rules in R_E and R_T respectively, generated by equations d such that $e \succ_{mod} d$.

The following two lemmas hold easily by construction.

Lemma 4. *Let $l \cdot \delta \to r \cdot \delta$ be a rule in R_T. Then $r \cdot \delta \equiv (l \cdot \delta)^x$, $l \cdot \delta$ is not marked on top, and the rule has been generated by an equation of the form $t\theta \simeq \perp$, where $t \in T$, $t\theta \in T(\mathcal{F})$, and $\mathit{Forget}(l \cdot \delta) \equiv t\theta$.*

Proof. We proceed by induction on the equation which has generated the rule w.r.t. $\succ_{mod}$. Observe that all equations e in T_{G} are of the form $t\theta \simeq \perp$, where $t \in T$ and $t\theta$ is a mark free term. Therefore $t\theta$ is not marked on top, and obviously $\mathit{Forget}(t\theta) \equiv t\theta$. Thus $t\theta \to (t\theta)^x$ fulfills the property.

Now let $u \cdot \rho$ denote the normal form of $t\theta$ w.r.t. R_T^e. We will show that the property is preserved in $u \cdot \rho \to (u \cdot \rho)^x$. By induction hypothesis, all rules $l' \cdot \delta' \to r' \cdot \delta'$ in R_T^e fulfill $\mathit{Forget}(l' \cdot \delta') \equiv \mathit{Forget}(r' \cdot \delta')$. Therefore, we necessarily have that $\mathit{Forget}(u \cdot \rho) \equiv t\theta$. We conclude by showing that in the normalization of $t \cdot \theta$ w.r.t. R_T^e, no step can be at topmost position, and therefore $u \cdot \rho$ is not marked on top: assume the contrary and let $l' \cdot \delta' \to r' \cdot \delta'$ be the rule in R_T^e, generated by an equation $d = (s \simeq \perp)$, which has been applied at topmost position; then $\mathit{Forget}(l' \cdot \delta') \equiv t\theta$, contradicting $e \succ_{mod} d$ since, by induction hypothesis again, $\mathit{Forget}(l' \cdot \delta') \equiv s$. ☐

Lemma 5. *For all rules $l \cdot \delta \to r \cdot \delta$ in R_{ET} we have that*

1. *$r \cdot \delta$ is irreducible by R_{ET} at non-topmost positions, and*
2. *$l \cdot \delta$ is irreducible by $R_{ET} \setminus \{l \cdot \delta \to r \cdot \delta\}$.*

Proof. For property 1, by construction, if an instance e generates $l \cdot \delta \to r \cdot \delta$, then the marked term $r \cdot \delta$ is irreducible by $R_E^e \cup R_T^e$ at non-topmost positions: this is clearly the case if $l \cdot \delta \to r \cdot \delta \in R_E$, since irreducibility w.r.t. $R_E^e \cup R_T^e$ of $r \cdot \delta$ at non-topmost positions is required, and also if $l \cdot \delta \to r \cdot \delta \in R_T$, since in this case $r \cdot \delta$ is irreducible by R_T^e and irreducibility w.r.t. R_E^e is required. We also have that $l \cdot \delta \to r \cdot \delta$ itself cannot reduce $r \cdot \delta$: if $l \cdot \delta \to r \cdot \delta \in R_E$ then it is not possible since $\mathit{Forget}(l \cdot \delta) \succ \mathit{Forget}(r \cdot \delta)$ and $\succ$ is irreflexive and fulfills the subterm property; if $l \cdot \delta \to r \cdot \delta \in R_T$ then, from Lemma 4, it follows that the marks in $l \cdot \delta$ and $r \cdot \delta$ do not coincide, and hence the reduction is neither possible. Finally, for every rule $l' \cdot \delta' \to r' \cdot \delta'$ generated by an instance d with $d \succ_{mod} e$, we must have that $\mathit{Forget}(l' \cdot \delta') \succeq \mathit{Forget}(l \cdot \delta)$. Now, if $l \cdot \delta \to r \cdot \delta \in R_E$ then, since $\mathit{Forget}(l \cdot \delta) \succ \mathit{Forget}(r \cdot \delta)$, we also have that $\mathit{Forget}(l' \cdot \delta') \succ \mathit{Forget}(r \cdot \delta)$, which implies that $l' \cdot \delta'$ cannot occur in $r \cdot \delta$. If $l \cdot \delta \to r \cdot \delta \in R_T$ then, by Lemma 4, $\mathit{Forget}(l \cdot \delta) \equiv \mathit{Forget}(r \cdot \delta)$ and therefore, if $\mathit{Forget}(l' \cdot \delta') \succ \mathit{Forget}(l \cdot \delta)$, we have that $l' \cdot \delta'$ cannot occur in $r \cdot \delta$ either and, if $\mathit{Forget}(l' \cdot \delta') \equiv \mathit{Forget}(l \cdot \delta)$, then $l' \cdot \delta'$ can only occur at topmost position in $r \cdot \delta$.

For property 2, by construction, if an instance e generates $l \cdot \delta \to r \cdot \delta$, then either irreducibility of $l \cdot \delta$ w.r.t. $R_E^e \cup R_T^e$ is required, or $l \cdot \delta$ is irreducible w.r.t. R_T^e and irreducibility w.r.t. R_E^e is required. Now, as before, for every rule $l' \cdot \delta' \to r' \cdot \delta'$ generated by an instance d with $d \succ_{mod} e$, we must have that $\mathit{Forget}(l' \cdot \delta') \succeq \mathit{Forget}(l \cdot \delta)$. If $\mathit{Forget}(l' \cdot \delta') \succ \mathit{Forget}(l \cdot \delta)$, then $l' \cdot \delta'$ cannot occur in $l \cdot \delta$. If, otherwise, $\mathit{Forget}(l' \cdot \delta') \equiv \mathit{Forget}(l \cdot \delta)$ then, since $l' \cdot \delta'$ must be irreducible by $R_E^d \cup R_T^d$, which contains $l \cdot \delta \to r \cdot \delta$, we have that $l' \cdot \delta' \not\equiv l \cdot \delta$, and hence $l' \cdot \delta'$ does not occur in $l \cdot \delta$ either. □

From the previous lemmas, convergence of R_{ET} follows.

Lemma 6. *R_{ET} is convergent.*

Proof. Recall that if $l \cdot \delta \to r \cdot \delta \in R_E$, then $\mathit{Forget}(l \cdot \delta) \succ \mathit{Forget}(r \cdot \delta)$ and, if $l \cdot \delta \to r \cdot \delta \in R_T$, then $l \cdot \delta \succ_m r \cdot \delta$ by Lemma 4. Now let $\succ_F$ denote the well-founded ordering defined as $l \cdot \delta \succ_F r \cdot \delta$ if, and only if, $\mathit{Forget}(l \cdot \delta) \succ \mathit{Forget}(r \cdot \delta)$, and let $\equiv_F$ denote the equivalence relation defined as $l \cdot \delta \equiv_F r \cdot \delta$ if, and only if, $\mathit{Forget}(l \cdot \delta) \equiv \mathit{Forget}(r \cdot \delta)$.

We will show that $\succ_F \cup \succ_m$ is a well-founded ordering: in every sequence $s_1 \succ_F \cup \succ_m s_2 \cdots$ there can only be finitely steps with $\succ_F$, since $s_i \cdot \gamma_i \succ_m s_j \cdot \gamma_j$ implies $s_i \cdot \gamma_i \equiv_F s_j \cdot \gamma_j$, and $\succ_F$ is well-founded and compatible with $\equiv_F$; then, from some point on, all steps must be with $\succ_m$ but, since $\succ_m$ is well-founded by Lemma 2, the sequence must be finite.

Therefore, $\succ_F \cup \succ_m$ is well-founded and, since rules in R_{ET} are oriented by $\succ_F \cup \succ_m$ and, moreover, by Lemma 5 their right-hand sides are irreducible by

R_{ET} at non-topmost positions, termination follows from Lemma 1. For confluence, since R_{ET} is terminating, by Newman's lemma we only need to show that R_{ET} is locally confluent, which holds by the second property of Lemma 5. $\square$

The same irreducibility and convergence results of R_{ET} hold as well for $Forget(R_E)$. In order to prove these, some auxiliary lemmas are given before.

Lemma 7. *Let $s \cdot \gamma$ be a marked subterm occurring in R_{ET}. If $s \cdot \gamma$ is marked on top, then $Forget(s \cdot \gamma)$ is an instance of a term in T.*

Proof. For all marked terms $(s \cdot \gamma)^x$ occurring in R_T, by construction, we have that $(s \cdot \gamma)^x$ occurs as a right-hand side of some rule in R_T. So in this case the property follows from Lemma 4. W.r.t. the marked terms occurring in R_E, the property follows from the facts that (i) the rules in R_E are generated by equations in $gnd_T(E)$ and (ii) for every marked subterm $(s \cdot \gamma)^x$ occurring in E, we have that $Forget(s \cdot \gamma)$ is an instance of a term in T. $\square$

Lemma 8. *Let $s \cdot \gamma$ be a marked subterm occurring in R_{ET}. If $Forget(s \cdot \gamma)$ is an instance of a term in T, then there is a rule of the form $t \cdot \theta \to (t \cdot \theta)^x$ in R_T, where $t \cdot \theta$ is not marked on top, and $s \cdot \gamma \equiv t \cdot \theta$ or $s \cdot \gamma \equiv (t \cdot \theta)^x$.*

Proof. We proceed by induction on $s \cdot \gamma$ w.r.t. $\rhd$. Let $l \cdot \delta \to r \cdot \delta$ be a rule in R_{ET} such that $l \cdot \delta \unrhd s \cdot \gamma$ or $r \cdot \delta \unrhd s \cdot \gamma$, and $Forget(s \cdot \gamma)$ is an instance of term t in T. Then there is an equation e of the form $t\theta \simeq \bot$ in T_{G}, for some ground substitution θ such that $t\theta \equiv Forget(s \cdot \gamma)$. Now, let $t' \cdot \theta'$ denote the normal form of $t\theta$ w.r.t. R^e_T. Recall that, by Lemma 4, we have that $Forget(s \cdot \gamma) \equiv Forget(t' \cdot \theta')$. Moreover, $t' \cdot \theta'$ is not marked on top since, by definition of $\succ_{mod}$ and Lemma 4, no equation smaller than e can generate a rule $l' \cdot \delta' \to r' \cdot \delta'$ in R_T with $Forget(l' \cdot \delta') \equiv t\theta$.

Now we show that either $s \cdot \gamma \equiv (t' \cdot \theta')^x$ or $s \cdot \gamma \equiv t' \cdot \theta'$. Assume the contrary. Then, since $Forget(s \cdot \gamma) \equiv Forget(t' \cdot \theta')$, there must be some position $p \neq \lambda$ such that either p is a marked position of $s \cdot \gamma$ and not of $t' \cdot \theta'$, or p is a marked position of $t' \cdot \theta'$ and not of $s \cdot \gamma$. Assume w.l.o.g. that p is such an innermost position.

1. If p is a marked position of $t' \cdot \theta'$ then, by construction of R_T and Lemma 4, we have that there is an equation of the form $t'' \cdot \theta'' \to (t'' \cdot \theta'')^x$ in R_T, where $t'' \cdot \theta''$ is not marked on top and $t' \cdot \theta'|_p \equiv (t'' \cdot \theta'')^x$. Now, if p is a non-marked position of $s \cdot \gamma$, this is a contradiction with Lemma 5 for $l \cdot \delta \to r \cdot \delta$.
2. If p is a marked position of $s \cdot \gamma$ then, by Lemma 7 and induction hypothesis, there is an equation of the form $s' \cdot \gamma' \to (s' \cdot \gamma')^x$ in R_T, where $s' \cdot \gamma'$ is not marked on top and $s \cdot \gamma|_p \equiv (s' \cdot \gamma')^x$. Now, if p is a non-marked position of $t' \cdot \theta'$, then $t' \cdot \theta'|_p \equiv s' \cdot \gamma'$. But this contradicts that $t' \cdot \theta'$ is in normal form w.r.t. R^e_T.

Finally, we show that $t' \cdot \theta' \to (t' \cdot \theta')^x \in R_T$. If $t' \cdot \theta' \to (t' \cdot \theta')^x \notin R_T$, then it must be because $t' \cdot \theta'$ is reducible by R^e_E. This reduction must take place necessarily at some position $p \neq \lambda$ since, by definition of $\succ_{mod}$, no equation generating a rule in R_E and containing $t' \cdot \theta'$ can be smaller than e. But, since

$s\cdot\gamma \equiv (t'\cdot\theta')^x$ or $s\cdot\gamma \equiv t'\cdot\theta'$, this is a contradiction with Lemma 5 for $l\cdot\delta \to r\cdot\delta$. □

Lemma 9. *Let $s \cdot \gamma$ be a marked subterm occurring, not as right-hand side, in some rule of R_E. Then $s \cdot \gamma$ is marked on top if, and only if, Forget$(s \cdot \gamma)$ is an instance of a term in T.*

Proof. Let $s \cdot \gamma$ be a marked term such that $l \cdot \delta \trianglerighteq s \cdot \gamma$ or $r \cdot \delta \triangleright s \cdot \gamma$ for some rule $l \cdot \delta \to r \cdot \delta$ in R_E. The left-to-right implication holds by Lemma 7. For the right-to-left implication, if *Forget*$(s \cdot \gamma)$ is an instance of a term in T then, by Lemma 8, there is a rule of the form $t \cdot \theta \to (t \cdot \theta)^x$ in R_T, where $t \cdot \theta$ is not marked on top and $s \cdot \gamma \equiv t \cdot \theta$ or $s \cdot \gamma \equiv (t \cdot \theta)^x$. Therefore, we necessarily have that $s \cdot \gamma \equiv (t \cdot \theta)^x$, since $s \cdot \gamma \equiv t \cdot \theta$ is a contradiction with Lemma 5 for $l \cdot \delta \to r \cdot \delta$. □

Then, from Lemmas 5 and 9, we obtain the same irreducibility result for *Forget*(R_E) as for R_{ET}, which easily implies the convergence of *Forget*(R_E).

Lemma 10. *For all rules $l \to r$ in Forget(R_E) we have that*

1. *r is irreducible by Forget(R_E) at non-topmost positions, and*
2. *l is irreducible by Forget$(R_E) \setminus \{l \to r\}$.*

Lemma 11. *Forget(R_E) is convergent.*

Proof. For all rules $l \cdot \delta \to r \cdot \delta$ in R_E we have that *Forget*$(l \cdot \delta) \succ$ *Forget*$(r \cdot \delta)$. Then, analogously to Lemma 6, we can conclude by Lemmas 10 and 1. □

Finally, we show that *Forget*$(R_E)^*$ is a model of *Forget*(E) whenever the pair $\langle E, T\rangle$ is saturated with respect to $\mathcal{E}$. Once we have that *Forget*(R_E) is convergent, the proof is a straightforward adaptation of the one in [BG94,NR01], but where the induction is done w.r.t. $\succ_i^{mul}$, taking $R =$ *Forget*(R_E) (see Sect. 3.1 for the definition of $\succ_i$). A contradiction is derived from the existence of a minimal w.r.t. $\succ_i^{mul}$ ground instance $e\theta$ in $gnd_T(E)$ of an equation e in E, such that $R^* \not\models$ *Forget*$(e\theta)$. As usual, we have that $e\theta$ has not generated a rule in R_E. If this is because $e\theta$ is reducible by R_E, then we have an inference by paramodulation and, if $e\theta$ is reducible by R_T, then we have an inference by marking and, in both cases, a contradiction is obtained from minimality of $e\theta$ and saturatedness of $\langle E, T\rangle$.

Theorem 1. *If a pair $\langle E, T\rangle$ is saturated with respect to $\mathcal{E}$, then Forget$(R_E)^* \models$ Forget(E).*

Proof. We proceed by induction w.r.t. $\succ_i^{mul}$, taking $R =$ *Forget*(R_E). A contradiction is derived from the existence of a minimal w.r.t. $\succ_i^{mul}$ ground instance e in $gnd_T(E)$ of the form $(s \cdot \gamma)\theta \simeq (t \cdot \gamma)\theta$ of an equation $s \cdot \gamma \simeq t \cdot \gamma$ in E, such that $R^* \not\models$ *Forget*(e).

Since $R^* \not\models$ *Forget*(e), then necessarily *Forget*$((s\cdot\gamma)\theta) \not\equiv$ *Forget*$((t\cdot\gamma)\theta)$, and hence we can assume w.l.o.g. that *Forget*$((s \cdot \gamma)\theta) \succ$ *Forget*$((t \cdot \gamma)\theta)$. Moreover, again since $R^* \not\models$ *Forget*(e), the equation e has not generated any rule of R_E. We have the following possibilities:

1. There is an instance of an equation $l \cdot \delta \simeq r \cdot \delta$ in $E \cup T_{\mathrm{E}}$ that has generated a rule $l' \cdot \delta' \to r' \cdot \delta'$ in R_{ET}, which reduces $(s \cdot \gamma)\theta$, or $(t \cdot \gamma)\theta$ at a non-topmost position.
   (a) $(s \cdot \gamma)\theta|_p \equiv l' \cdot \delta'$ for some position p.
      i. p is a non-variable position of *Forget*$(s \cdot \gamma)$.
         A. $l \cdot \delta \simeq r \cdot \delta \in T_{\mathrm{E}}$. Then $l\delta \equiv l \in T$ and $r\delta \equiv r \equiv \bot$. Moreover $l' \cdot \delta' \to r' \cdot \delta' \in R_T$ and, by Lemma 4, we have that $r' \cdot \delta' \equiv (l' \cdot \delta')^x$, $l' \cdot \delta'$ is not marked on top, and *Forget*$(l' \cdot \delta')$ is a ground instance of l. Since $l' \cdot \delta'$ is not marked on top and $(s \cdot \gamma)\theta|_p \equiv l' \cdot \delta'$, then p is a non-marked position of $(s \cdot \gamma)\theta$, and hence of $s \cdot \gamma$. Therefore there is an inference by *marking*,

$$\frac{s \cdot \gamma \simeq t \cdot \gamma}{(s \cdot \gamma[(s \cdot \gamma|_p)^x]_p \simeq t \cdot \gamma)\sigma},$$

         where $\sigma = mgu(\mathit{Forget}(s \cdot \gamma|_p), l)$, whose conclusion has an instance d of the form $(s \cdot \gamma[(s \cdot \gamma|_p)^x]_p \simeq t \cdot \gamma)\theta$. Moreover, since p is a non-marked position of $s \cdot \gamma$, we have that either $s \gtrdot s'$, or $s \doteq s'$ and $\mathit{Ran}(\gamma) \succ_m^{mul} \mathit{Ran}(\gamma')$, where $s' \cdot \gamma'$ denotes $s \cdot \gamma[(s \cdot \gamma|_p)^x]_p$. Therefore, $s \cdot \gamma \succ_i s \cdot \gamma[(s \cdot \gamma|_p)^x]_p$ either by case (i) or (ii), and hence $e \succ_i^{mul} d$. Now, since $\langle E, T\rangle$ is saturated w.r.t. $\mathcal{E}$, this inference is redundant in E. Therefore, since $e \in gnd_T(E)$, we have that $\mathit{Forget}(gnd_T(E)^{\prec_{red}^{mul} e}) \cup \mathit{Forget}(gnd_T(E)^{\preceq_{red}^{mul} d}) \models \mathit{Forget}(d)$. Moreover, as seen before, we have $e \succ_i^{mul} d$, and $\succ_{red} \subseteq \succ_i$. Therefore since, by minimality of e, we have that $R^* \models \mathit{Forget}(gnd_T(E)^{\prec_i^{mul} e})$, then also $R^* \models \mathit{Forget}(gnd_T(E)^{\prec_{red}^{mul} e}) \cup \mathit{Forget}(gnd_T(E)^{\preceq_{red}^{mul} d})$ and hence $R^* \models \mathit{Forget}(d)$, contradicting $R^* \not\models \mathit{Forget}(e)$.
         B. $l \cdot \delta \simeq r \cdot \delta \in E$. Then $l' \cdot \delta' \to r' \cdot \delta' \in R_E$, with $l' \cdot \delta' \equiv (l \cdot \delta)\theta$, $r' \cdot \delta' \equiv (r \cdot \delta)\theta$, *Forget*$((l \cdot \delta)\theta) \succ$ *Forget*$((r \cdot \delta)\theta)$, and moreover $l \succ_r r$ or $r \cdot \delta$ is marked on top. Therefore there is an inference by *paramodulation*,

$$\frac{l \cdot \delta \simeq r \cdot \delta \qquad s \cdot \gamma \simeq t \cdot \gamma}{(s \cdot \gamma[r \cdot \delta]_p \simeq t \cdot \gamma)\sigma},$$

         where $\sigma = mgu(s \cdot \gamma|_p, l \cdot \delta)$, whose conclusion has an instance d of the form $(s \cdot \gamma[r \cdot \delta]_p \simeq t \cdot \gamma)\theta$. As before, we have that $e \succ_i^{mul} d$:
         – If $r \cdot \delta$ is marked on top, then either p is a marked position of $s \cdot \gamma$ or else, as seen in the previous case, $(s \cdot \gamma) \succ_i (s \cdot \gamma[(s \cdot \gamma|_p)^x]_p)$. Therefore, in this particular case, $(s \cdot \gamma)\theta \succeq_i (s \cdot \gamma[(s \cdot \gamma|_p)^x]_p)\theta$. Moreover, since $(s \cdot \gamma)\theta|_p \equiv (l \cdot \delta)\theta$, we have $(s \cdot \gamma[(s \cdot \gamma|_p)^x]_p)\theta \equiv (s \cdot \gamma[(l \cdot \delta)^x]_p)\theta$. And, since *Forget*$((l \cdot \delta)\theta) \to_R$ *Forget*$((r \cdot \delta)\theta)$, we have $(s \cdot \gamma[(l \cdot \delta)^x]_p)\theta \succ_i (s \cdot \gamma[(r \cdot \delta)^x]_p)\theta$ by case (iii). Therefore, $(s \cdot \gamma)\theta \succeq_i (s \cdot \gamma[(l \cdot \delta)^x]_p)\theta \succ_i (s \cdot \gamma[(r \cdot \delta)^x]_p)\theta$, and thus $e \succ_i^{mul} d$.

– If $l \succ_r r$, then either there is no prefix p' of p such that p' is a marked position of $s \cdot \gamma$, and then, by monotonicity and stability under substitutions of $\succ_r$, $(s \cdot \gamma)\theta \succ_i (s \cdot \gamma[r \cdot \delta]_p)\theta$ by case (i), or else $(s \cdot \gamma)\theta \succ_i (s \cdot \gamma[r \cdot \delta]_p)\theta$ by case (iii), and hence $e \succ_i^{mul} d$.

Now, since $\langle E, T\rangle$ is saturated w.r.t. $\mathcal{E}$, this inference is redundant in E. Then, as in case 1(a)iA, we can prove that $R^* \models$ *Forget*(d) which, together with the fact that $R^* \models$ *Forget*$((l \cdot \delta \simeq r \cdot \delta)\theta)$, contradicts $R^* \not\models$ *Forget*(e).

ii. Lifting: *Forget*$(s \cdot \gamma)|_{p'}$ is a variable x for some prefix p' of p. Then $p = p' \cdot p''$ for some position p'' such that $x\theta|_{p''} \equiv l' \cdot \delta'$. Now let θ' be a ground substitution with the same domain as θ but where $x\theta' \equiv x\theta[r' \cdot \delta']_{p''}$ and $y\theta' \equiv y\theta$ for all other variables y. Then, by the same arguments as in case 1(a)i, we can argue that $e \succ_i^{mul} (s \cdot \gamma \simeq t \cdot \gamma)\theta'$. Now, since $l' \cdot \delta' \to r' \cdot \delta' \in R_{ET}$, we have that either $l' \cdot \delta' \to r' \cdot \delta' \in R_E$, or else $l' \cdot \delta' \to r' \cdot \delta' \in R_T$ and, by Lemma 4, *Forget*$(l' \cdot \delta') \equiv$ *Forget*$(r' \cdot \delta')$. Therefore, since $R =$ *Forget*(R_E), we have that $R^* \models$ *Forget*$(l' \cdot \delta' \simeq r' \cdot \delta')$. And, since $R^* \not\models$ *Forget*(e), we necessarily have that $R^* \not\models$ *Forget*$((s \cdot \gamma \simeq t \cdot \gamma)\theta')$. We also have that $(s \cdot \gamma \simeq t \cdot \gamma)\theta' \in gnd_T(E)$, since $e \in gnd_T(E)$ and, moreover, for every marked subterm of the form $(w \cdot \varphi)^z$ in $r' \cdot \delta'$ we have that *Forget*$(w \cdot \varphi)$ is an instance of a term in T by Lemma 7. Therefore $(s \cdot \gamma \simeq t \cdot \gamma)\theta'$ contradicts the minimality of e.

(b) $(t \cdot \gamma)\theta|_p \equiv l' \cdot \delta'$ for some position $p \neq \lambda$.

i. p is a non-variable position of *Forget*$(t \cdot \gamma)$. We proceed analogously to case 1(a)i, with the only difference that the inferences do not take place at topmost position.

ii. *Forget*$(t \cdot \gamma)|_{p'}$ is a variable x for some prefix p' of p. We proceed analogously to case 1(a)ii.

2. $s \not\succ_r t$ and $t \cdot \gamma$ is not marked on top. In this case we can apply the *mark detection* rule

$$T \Longrightarrow T \cup \{Forget(t \cdot \gamma)\}.$$

Now, since $\langle E, T\rangle$ is saturated w.r.t. $\mathcal{E}$, this inference is redundant in T. Therefore *Forget*$(t \cdot \gamma)$ is an instance of a term t' in T. But then there is an inference by *marking*,

$$\frac{s \cdot \gamma \simeq t \cdot \gamma}{s \cdot \gamma \simeq (t \cdot \gamma)^x}.$$

Note that, if t is a variable, then t' is also a variable. Now, since $t \cdot \gamma$ is not marked on top, we clearly have that $t \cdot \gamma \succ_i (t \cdot \gamma)^x$ by case (i). From this, a contradiction is derived as in case 1(a)iA. □

5 Horn Clauses

In this section we generalize the results of the previous section to Horn clauses. In the inference system it is assumed that in each clause with a non-empty antecedent one of these negative equations, the one that is written underlined, has been *selected* (c.f. [BG98]).

Definition 8. *Let S be a set of Horn clauses and T be a set of terms, which satisfy the following condition: for every marked subterm $(s \cdot \gamma)^x$ occurring in S, we have that Forget$(s \cdot \gamma)$ is an instance of a term in T. Our inference system $\mathcal{H}$ consists of the following five inference rules and a mark detection rule:*

Paramodulation right*:*

$$\frac{\to l \cdot \delta \simeq r \cdot \delta \qquad \to s \cdot \gamma \simeq t \cdot \gamma}{(\to s \cdot \gamma[r \cdot \delta]_p \simeq t \cdot \gamma)\sigma} \qquad \textit{if}$$

1. *$\sigma = mgu(l \cdot \delta, s \cdot \gamma|_p)$, the most general unifier of $l \cdot \delta$ and $s \cdot \gamma|_p$,*
2. *Forget$(s \cdot \gamma|_p)$ is not a variable,*
3. *$l \succ_r r$ or $r \cdot \delta$ is marked on top, and*
4. *for some ground substitution θ, we have Forget$((l \cdot \delta)\sigma\theta) \succ$ Forget$((r \cdot \delta)\sigma\theta)$ and, if $p = \lambda$, then we also have Forget$((s \cdot \gamma)\sigma\theta) \succ$ Forget$((t \cdot \gamma)\sigma\theta)$.*

Paramodulation left:

$$\frac{\to l \cdot \delta \simeq r \cdot \delta \qquad \Gamma, \underline{s \cdot \gamma \simeq t \cdot \gamma} \to \Delta}{(\Gamma, s \cdot \gamma[r \cdot \delta]_p \simeq t \cdot \gamma \to \Delta)\sigma} \qquad \textit{if}$$

1. *$\sigma = mgu(l \cdot \delta, s \cdot \gamma|_p)$,*
2. *Forget$(s \cdot \gamma|_p)$ is not a variable,*
3. *$l \succ_r r$ or $r \cdot \delta$ is marked on top, and*
4. *for some ground substitution θ, we have Forget$((l \cdot \delta)\sigma\theta) \succ$ Forget$((r \cdot \delta)\sigma\theta)$ and, if $p = \lambda$, then we also have Forget$((s \cdot \gamma)\sigma\theta) \succ$ Forget$((t \cdot \gamma)\sigma\theta)$.*

Marking right:

$$\frac{\to s \cdot \gamma[u \cdot \delta]_p \simeq t \cdot \gamma}{(\to s \cdot \gamma[(u \cdot \delta)^x]_p \simeq t \cdot \gamma)\sigma} \qquad \textit{if}$$

1. *$\sigma = mgu($Forget$(u \cdot \delta), u')$ for some term u' in T,*
2. *u is not a variable, or u' is a variable and $p = \lambda$, and*
3. *x is a fresh variable.*

Marking left:

$$\frac{\Gamma, \underline{s \cdot \gamma[u \cdot \delta]_p \simeq t \cdot \gamma} \to \Delta}{(\Gamma, s \cdot \gamma[(u \cdot \delta)^x]_p \simeq t \cdot \gamma \to \Delta)\sigma} \qquad \textit{if}$$

1. *$\sigma = mgu($Forget$(u \cdot \delta), u')$ for some term u' in T,*
2. *u is not a variable,*
3. *x is a fresh variable, and*
4. *if $p = \lambda$ then, for some substitution θ, we have that Forget$((s \cdot \gamma)\sigma\theta) \succ$ Forget$((t \cdot \gamma)\sigma\theta)$.*

Equality resolution:

$$\frac{\Gamma, \underline{s \cdot \gamma \simeq t \cdot \gamma} \to \Delta}{(\Gamma \to \Delta)\sigma} \qquad \textit{if } \sigma = mgu(s \cdot \gamma, t \cdot \gamma).$$

Mark detection:

$$T \Longrightarrow T \cup \{\mathit{Forget}(r \cdot \delta)\} \qquad \textit{if}$$

1. $\to l \cdot \delta \simeq r \cdot \delta \in S$,
2. $l \not\succ_r r$ *and* $r \cdot \delta$ *is not marked on top, and*
3. *for some ground substitution* θ*, we have that* $\mathit{Forget}((l{\cdot}\delta)\theta) \succ \mathit{Forget}((r{\cdot}\delta)\theta)$.

Definition 9. *Let* S *be a set of Horn clauses. By* E_S *we denote the set of equations occurring as positive unit clauses in* S*, that is,* $E_S = \{s \cdot \gamma \simeq t \cdot \gamma \mid \to s \cdot \gamma \simeq t \cdot \gamma \in S\}$*, and we denote by* R_{ST} *the set of rules generated by* E_S *and* T *as in Definition 7 of Sect. 4.2 (i.e.* $R_{ST} = R_{E_S T}$*). Also, following the notation of Definition 7, we denote by* R_S *and* R_T *the set of rules generated by all equations in* $gnd_T(E_S)$ *and* T_{G}*, respectively.*

We use standard multiset extensions to lift the orderings on marked terms to orderings on marked equations and marked clauses: Let C be a clause and let $\mathit{emul}(s \cdot \gamma \simeq t \cdot \gamma)$ be $\{s \cdot \gamma, t \cdot \gamma\}$ if $s \cdot \gamma \simeq t \cdot \gamma$ is a positive equation in C, and $\{s{\cdot}\gamma, s{\cdot}\gamma, t{\cdot}\gamma, t{\cdot}\gamma\}$ if it is negative. Then, if $\succ$ is an ordering on marked terms, we define the ordering $\succ^e$ on marked equations by $e \succ^e e'$ if $\mathit{emul}(e) \succ^{mul} \mathit{emul}(e')$; similarly, $\succ^c$ on clauses is defined by $C \succ^c D$ if $\mathit{mse}(C) \;\; (\succ^{mul})^{mul} \;\; \mathit{mse}(D)$, where $\mathit{mse}(C)$ is the multiset of all $\mathit{emul}(e)$ for occurrences e of equations in C. (We analogously lift every equivalence relation $\sim$ on marked terms to an equivalence relation $\sim^e$ on equations and $\sim^c$ on clauses.)

Then the notions of redundancy and saturation are directly obtained from the ones given in Sect. 4.1 by using Horn clauses instead of equations and the inference system $\mathcal{H}$ instead of $\mathcal{E}$, and lifting the orderings adequately.

Using Lemmas 9, 10 and 11, and Theorem 1 applied to the rewrite model obtained from the positive unit clauses, the following completeness theorem is obtained in the usual way.

Theorem 2. *(Completeness) Let* S *be a set of Horn clauses and* T *a set of terms such that the pair* $\langle S, T\rangle$ *is saturated with respect to* $\mathcal{H}$*. Then* $\Box \in S$ *if, and only if,* $\mathit{Forget}(S)$ *is unsatisfiable.*

Proof. The left-to-right implication is trivial. For the other one, as in Theorem 1, we proceed by induction w.r.t. $\succ_i^c$, taking $R = \mathit{Forget}(R_S)$.

Again it is proved that R^* is a model of $\mathit{Forget}(S)$ by deriving a contradiction from the existence of a minimal w.r.t. $\succ_i^c$ ground instance $C\theta$ in $gnd_T(S)$ of a clause C in S such that $R^* \not\models \mathit{Forget}(C\theta)$. Since $\langle S, T\rangle$ is saturated w.r.t. $\mathcal{H}$, then $\langle E_S, T\rangle$ is saturated w.r.t. $\mathcal{E}$ and, by Theorem 1, we have that $R^* \models \mathit{Forget}(E_S)$. Therefore C cannot be a positive unit clause. Two cases have to be considered:

1. C is a clause $\Gamma, \underline{s \cdot \gamma \simeq t \cdot \gamma} \rightarrow \Delta$, where $\mathit{Forget}((s \cdot \gamma)\theta) \equiv \mathit{Forget}((t \cdot \gamma)\theta)$. Then there is an inference by *equality resolution*,

$$\frac{\Gamma, \underline{s \cdot \gamma \simeq t \cdot \gamma} \rightarrow \Delta}{(\Gamma \rightarrow \Delta)\sigma},$$

where $\sigma = mgu(s \cdot \gamma, t \cdot \gamma)$. Let D denote the clause $(\Gamma \rightarrow \Delta)\sigma$. Now observe that $C\theta \succ_i^c D\theta$. From this fact and from the saturatedness assumption, a contradiction is derived as in Theorem 1: Since $\langle S, T \rangle$ is saturated w.r.t. $\mathcal{H}$, this inference is redundant in S. Therefore, since $C\theta \in gnd_T(S)$, we have that $\mathit{Forget}(gnd_T(S)^{\prec_{red}^{mul} C\theta}) \cup \mathit{Forget}(gnd_T(S)^{\preceq_{red}^{c} D\theta}) \models \mathit{Forget}(D\theta)$. Moreover, by minimality of $C\theta$, we have that $R^* \models \mathit{Forget}(gnd_T(S)^{\prec_i^c C\theta})$. But then, since $C\theta \succ_i^c D\theta$ and $\succ_{red}^c \subseteq \succ_i^c$, we have that $R^* \models \mathit{Forget}(D\theta)$, contradicting $R^* \not\models \mathit{Forget}(C\theta)$.
2. C is a clause $\Gamma, \underline{s \cdot \gamma \simeq t \cdot \gamma} \rightarrow \Delta$ where, w.l.o.g., $\mathit{Forget}((s \cdot \gamma)\theta) \succ \mathit{Forget}((t \cdot \gamma)\theta)$. Let s' denote $\mathit{Forget}((s \cdot \gamma)\theta)$ and t' denote $\mathit{Forget}((t \cdot \gamma)\theta)$. Then, since $R^* \not\models \mathit{Forget}(C\theta)$, we have that $R^* \models s' \simeq t'$ and, since R is convergent by Lemma 11, there must be a rewrite proof of $s' \simeq t'$ with R, that is, s' and t' must rewrite into the same normal form with R. This implies that either t' is reducible at a non-topmost position or else s' is reducible. (Note that it cannot be the case that the only possible reduction step on $s' \simeq t'$ is at the topmost position of t'. By such a step, a new term t'' is obtained with $t' \succ t''$ and t'' again irreducible at non-topmost positions by Lemma 10; since $s' \succ t'$, such a sequence of topmost steps on t' can never produce s'). We consider the case where s' is reducible. (The other one is analogous.) Then there is an instance of an equation $l \cdot \delta \simeq r \cdot \delta$ in E_S that has generated a rule $(l \cdot \delta)\theta \rightarrow (r \cdot \delta)\theta$ in R_S, such that $\mathit{Forget}((s \cdot \gamma)\theta)|_p \equiv \mathit{Forget}((l \cdot \delta)\theta)$ for some position p. We have the following possibilities:
   (a) $(s \cdot \gamma)\theta|_p \equiv (l \cdot \delta)\theta$. Then either the lifting argument applies like in Theorem 1, or else there is an inference by *paramodulation left*,

   $$\frac{\rightarrow l \cdot \delta \simeq r \cdot \delta \qquad \Gamma, \underline{s \cdot \gamma \simeq t \cdot \gamma} \rightarrow \Delta}{(\Gamma, s \cdot \gamma[r \cdot \delta]_p \simeq t \cdot \gamma \rightarrow \Delta)\sigma},$$

   which leads to a contradiction analogously to case 1(a)iB of Theorem 1.
   (b) $(s \cdot \gamma)\theta|_p \not\equiv (l \cdot \delta)\theta$. Then, since $\mathit{Forget}((s \cdot \gamma)\theta)|_p \equiv \mathit{Forget}((l \cdot \delta)\theta)$, there must be some positions q and p' such that $q = p \cdot p'$ and $\mathit{Forget}((s \cdot \gamma)\theta)|_q \equiv \mathit{Forget}((l \cdot \delta)\theta)|_{p'}$, and either q is a marked position of $(s \cdot \gamma)\theta$ and p' is a non-marked position of $(l \cdot \delta)\theta$, or q is a non-marked position of $(s \cdot \gamma)\theta$ and p' is a marked position of $(l \cdot \delta)\theta$.
      i. If q is a marked position of $(s \cdot \gamma)\theta$ then, since $C\theta \in gnd_T(S)$ and, moreover, for every marked term of the form $(l' \cdot \delta')^x$ in occurring in S we have that $\mathit{Forget}(l' \cdot \delta')$ is an instance of a term in T, we must have that $\mathit{Forget}((s \cdot \gamma)\theta|_q)$ is an instance of a term t in T. But then also $\mathit{Forget}((l \cdot \delta)\theta|_{p'})$ is an instance of t and, therefore, by Lemma 9 we have that $(l \cdot \delta)\theta|_{p'}$ is marked on top, contradicting that p' is a non-marked position of $(l \cdot \delta)\theta$.

ii. If q is a non-marked position of $(s \cdot \gamma)\theta$ and p' is a marked position of $(l \cdot \delta)\theta$ then,
   A. if $\mathit{Forget}(s \cdot \gamma)|_{q'}$ is a variable x for some prefix q' of q, we have that $q = q' \cdot q''$ for some position q'' such that q'' is a non-marked position of $x\theta$ and $\mathit{Forget}(x\theta)|_{q''} \equiv \mathit{Forget}((l \cdot \delta)\theta)|_{p'}$. Now let θ' be a ground substitution with the same domain as θ but where $x\theta' \equiv x\theta[(x\theta|_{q''})^z]_{q''}$ and $y\theta' \equiv y\theta$ for all other variables y. We have that $C\theta \succ_i^c C\theta'$ and $R^* \not\models \mathit{Forget}(C\theta')$. Moreover, we have that $C\theta' \in \mathit{gnd}_T(S)$, since $C\theta \in \mathit{gnd}_T(S)$ and $\mathit{Forget}((l \cdot \delta)\theta|_{p'})$ is an instance of a term in T by Lemma 7. Therefore $C\theta'$ contradicts the minimality of $C\theta$.
   B. Otherwise, there is an inference by *marking left*,

$$\frac{\Gamma, \underline{s \cdot \gamma \simeq t \cdot \gamma} \to \Delta}{(\Gamma, s \cdot \gamma[(s \cdot \gamma|_q)^x]_p \simeq t \cdot \gamma \to \Delta)\sigma},$$

   where $\sigma = \mathit{mgu}(\mathit{Forget}(s \cdot \gamma|_q), t)$, for the term t in T such that $\mathit{Forget}((l \cdot \delta)\theta|_{p'})$ is an instance of t, which leads to a contradiction analogously to case 1(a)iA of Theorem 1. □

6 Derivations and Redundancy of Clauses

In the previous section, we showed that an unsatisfiable saturated set of (Horn) clauses contains the empty clause. Saturatedness was defined in terms of redundancy of inferences. Here we address the problem of how to compute such saturated sets. Usually, a saturation procedure is modeled by means of a *derivation*, a sequence of sets of clauses where each set can be obtained from the previous either by adding a logical consequence or by removing some redundant clause. In our case, a ground clause C is defined to be *redundant* in a set of ground clauses G if $\mathit{Forget}(G^{\prec^c_{red} C}) \models \mathit{Forget}(C)$. Then a clause C is *redundant* in a set of clauses S if, for all sets of terms M and ground substitutions θ with marks in M, $C\theta$ is redundant in $\mathit{gnd}_M(S)$.

Since we deal with a set of clauses S and a set of terms T, the notion of derivation is adapted as follows: a *derivation* is a (possibly infinite) sequence of pairs of sets of clauses and terms $\langle S_0, T_0\rangle, \langle S_1, T_1\rangle, \ldots$ where for each $\langle S_{i+1}, T_{i+1}\rangle$ we have either

1. $T_{i+1} = T_i$ and
   (a) $S_{i+1} = S_i \cup \{C\}$, for some C such that $\mathit{Forget}(S_i) \models \mathit{Forget}(C)$, or
   (b) $S_{i+1} = S_i \setminus \{C\}$, for some C which is redundant in S_i, or
2. $S_{i+1} = S_i$ and
   (a) $T_{i+1} = T_i \cup \{t\}$ for some term t, or
   (b) $T_{i+1} = T_i \setminus \{t\}$ where t is an instance of a term in $T_i \setminus \{t\}$.

The abstract redundancy notions for inferences presented in Sect. 4.1 cover the so-called *forward* redundancy elimination techniques in which, for example, the conclusion of an inference need not be stored if it follows from smaller clauses. On the other hand, redundancy for clauses is intended to cover the so-called *backward* redundancy elimination techniques in which, for example, an existing clause can be deleted if it follows from smaller ones, including newer clauses that have been generated later on. It is well-known that simplification by rewriting fits into the notion of derivation [NR01]. As shown in the following section, we can add the simplified clause and then remove the original one, which has become redundant.

Clauses belonging, from some i on, to all S_k with $k > i$, are called *persistent*. The set of persistent clauses S_∞ is formally defined as $S_\infty = \cup_i \cap_{k>i} S_k$. Analogously, the set of persistent terms is defined as $T_\infty = \cup_i \cap_{k>i} T_k$. The usual property that states that all non-persistent clauses occurring in a derivation are redundant w.r.t. the persistent ones also holds here.

Lemma 12. *Let $\langle S_0, T_0\rangle, \langle S_1, T_1\rangle, \ldots$ be a derivation. Then, for every term t in $\cup_i T_i$, we have that t is an instance of a term in T_∞.*

Proof. Let t be a term in some T_i. By induction on the derivation, it follows that, for all T_j with $j > i$, we have that t is an instance of a term in T_j. Therefore t is also an instance of a term in T_∞. □

Lemma 13. *Let $\langle S_0, T_0\rangle, \langle S_1, T_1\rangle, \ldots$ be a derivation and let C be a clause in $(\cup_i S_i) \setminus S_\infty$. Then C is redundant in S_∞.*

Proof. We need to prove that, for all sets of terms M and ground substitutions θ with marks in M, $C\theta$ is redundant in $gnd_M(S_\infty)$. For this, we assume the contrary for some set of terms N and some substitution σ with marks in N, and derive a contradiction.

Since $C \in (\cup_i S_i)\setminus S_\infty$, there is some S_j such that C is redundant in S_j. Then, for all sets of terms M and substitutions θ with marks in M, $C\theta$ is redundant in $gnd_M(S_j)$, and hence in $gnd_M(\cup_i S_i)$. Therefore, in particular, $C\sigma$ is redundant in $gnd_N(\cup_i S_i)$.

Now let G be the minimal (w.r.t. $(\succ^c_{red})^{mul}$) subset of $gnd_N(\cup_i S_i)$ such that $C\sigma$ is redundant in G. Since $gnd_N(S_\infty) \nsubseteq G$ (because $C\sigma$ is not redundant in $gnd_N(S_\infty)$), there is some instance of the form $D\sigma'$ in G for some clause D which is not persistent, and hence redundant in some S_k. But then $Forget(G') \models Forget(D\sigma')$ for some $G' \subseteq gnd_N(S_k)^{\prec^c_{red} D\sigma'}$. Therefore, since $gnd_N(S_k)^{\prec^c_{red} D\sigma'} \subseteq gnd_N(\cup_i S_i)^{\prec^c_{red} D\sigma'}$, $C\sigma$ is redundant as well in the smaller subset $G' \cup G \setminus \{D\sigma'\}$, contradicting the minimality of G. □

In order to obtain a saturated set in the limit of a derivation, some notion of *fairness* of the derivation is required. We will use the following: a derivation $\langle S_0, T_0\rangle, \langle S_1, T_1\rangle, \ldots$ is *fair* with respect to an inference system $\mathcal{I}nf$ if (i) every inference using clauses in S_∞ and terms in T_∞ is redundant in some S_j and (ii) every mark detection rule using clauses in S_∞ is redundant in some T_j.

Lemma 14. *If $\langle S_0, T_0\rangle, \langle S_1, T_1\rangle, \ldots$ is a fair derivation with respect to an inference system $\mathcal{I}nf$, then $\langle S_\infty, T_\infty\rangle$ is saturated with respect to $\mathcal{I}nf$.*

Proof. By fairness, every inference π using clauses in S_∞ and terms in T_∞ is redundant in some S_j. Then, by Lemma 13, π is also redundant in S_∞. By fairness again, every mark detection rule using clauses in S_∞ is redundant in some T_j and hence, by Lemma 12, so is in T_∞. Therefore $\langle S_\infty, T_\infty \rangle$ is saturated. $\square$

Then, from Lemma 14 and Theorem 2 it follows the refutation completeness of any theorem proving procedure which computes a fair derivation w.r.t. the inference system $\mathcal{H}$. Note that, in order to ensure fairness in practice, in the inference systems presented in this paper it suffices that no inference (or mark detection rule) with persistent premises is postponed infinitely many times, since adding the conclusion always makes the inference (or mark detection rule) redundant. For example, this could be achieved by periodically considering all inferences with the clause whose *size* (in number of symbols) is smallest. If a certain clause persists then it will eventually be considered, since there are only finitely many clauses with smaller size.

In principle, it would be desirable to postpone the use of *marking* inference rules as much as possible, since marking of terms generally will not allow much demodulation. However, our redundancy notions do not preserve refutation completeness if blockings are not introduced at all, as shown in Example 4 of the following section.

7 Practical Notions of Redundancy

In this section we provide sufficient conditions for proving redundancy of inferences, which include simplification by rewriting for those rules included in the reduction ordering $\succ_r$.

Let S be a set of Horn clauses. By S_r we denote the set of rules $l \cdot \delta \to r \cdot \delta$ such that the clause $\to l \cdot \delta \simeq r \cdot \delta$ is in S and $l \succ_r r$.

An inference with premises $C_1, \ldots, C_n$ and conclusion D can be shown redundant in S in two steps:

1. First, D is simplified by rewriting it into D' using marked rules $l \cdot \delta \to r \cdot \delta$ in S_r. In order to ensure that we are only using smaller (w.r.t. $\succ^c_{red}$) instances of clauses in S, we have the following sufficient conditions. If D is of the form $\Gamma \to s \simeq t$, then (i) we can apply any rewriting step on Γ, but only at positions of the skeletons; (ii) we can apply any rewriting step at a non-top position of the skeleton of s (resp. t) obtaining s' (resp. t'), and after that we can apply any rewriting step at any position of the skeleton of s' (resp. t'); (iii) we can apply a rewriting step at top position of s (resp. t) obtaining s' (resp. t') if the instance of the rule applied is smaller (w.r.t. $\succ^c_{red}$) than C_n or smaller than or equal to D, and after that we can apply any rewriting step at any position of the skeleton of s' (resp. t'). Note that in all these cases we can show that for any ground instance of the inference, the ground instances of the applied rules are either smaller (w.r.t. $\succ^c_{red}$) than the instance of C_n or smaller than or equal to the instance of D.

2. Second, we check whether $D_1, \ldots, D_n \models D'$ for some instances of clauses in S s.t. $D'(\trianglerighteq_{sk} \cup \succ_m)^c D_i$ for all i in $1 \ldots n$, where $s \cdot \gamma \trianglerighteq_{sk} t \cdot \delta$ if $t \cdot \delta$ occurs in $s \cdot \gamma$ at some position of its skeleton. This includes, among other cases, the case where D' is in S, the case where there is a version of D' with additional marks in S, and the case where D' is a tautology. If this situation does not apply, then we can add D' to S to make the inference redundant.

Example 3. In Example 1 of the introduction we showed how, by an ordered paramodulation with $\underline{f(f(x))} \simeq f(g(f(x)))$ on $f(f(y)) \simeq f(g(\underline{f(y)}))$, we got the equation $f(f(f(x))) \simeq \overline{f(g(f(g(f(x)))))}$, i.e. $f(f(f(z))) \simeq f(\overline{g(f(g(f(z))))})$. In this case, $f(f(f(z)))$ could be rewritten in two steps with $f(f(x)) \to f(g(f(x)))$ into $f(g(f(g(f(z)))))$, and hence the conclusion of the inference could be simplified into a tautology. Observe that the first step is applied at a non-top position, obtaining $f(f(g(f(z)))$, and then the second step can be applied on top obtaining $f(g(f(g(f(z)))))$.

All previous results can be adapted in the usual way for showing redundancy of a clause C by using only smaller clauses w.r.t. $\prec^c_{red}$ than C.

In the following example, we show how demodulation w.r.t. $\succ_r$ and redundancy with smaller equations w.r.t. the subterm relation can cause incompleteness if no blockings are introduced at all. This thus shows the need for some control or, in other words, that our blockings are not merely a technicality to prove completeness.

Example 4. Let $\succ_r$ be a reduction ordering such that

$$f(x,x) \succ_r f(a,b) \text{ and}$$
$$f(x,x) \succ_r f(b,a).$$

Then necessarily

$$a \not\succ_r b \text{ and}$$
$$b \not\succ_r a.$$

Let S denote the following inconsistent set of Horn clauses:

1) $\to a \simeq b$
2) $\to f(x,x) \simeq f(a,b)$
3) $\to f(x,x) \simeq f(b,a)$
4) $\to f(x,x) \simeq f(b,b)$
5) $\to g(f(a,b)) \simeq g(g(f(b,a)))$
6) $x \simeq g(x) \to$

First of all, we will show how the empty clause can be derived under our inference system (which introduces the convenient blockings). Let $\succ$ be a west ordering which is an extension of $\succ_r$ and that includes

$$g(g(f(a,b))) \succ g(f(a,b)) \succ g(g(f(b,a))) \succ g(f(b,a))$$
$$\succ f(a,a) \succ f(b,b) \succ f(a,b) \succ f(b,a) \succ a \succ b.$$

Initially, the set T of terms that are going to be marked is empty. Then, since $a \succ b$ and $a \not\succ_r b$, the *mark detection* rule gives us $T = \{b\}$. After this, an inference by *marking right* into 1 gives us

$$7) \quad \rightarrow a \simeq b^{x_0}$$

Now, an inference by *paramodulation right* with 7 into 5 is possible (recall that an inference by paramodulation right with 1 into 5 is not possible since $a \simeq b$ does not fulfill $a \succ_r b$ and b is not marked on top), giving

$$8) \quad \rightarrow g(f(b^{x_1}, b)) \simeq g(g(f(b, a)))$$

A new inference by *paramodulation right* with 7 into 8 gives us

$$9) \quad \rightarrow g(f(b^{x_1}, b)) \simeq g(g(f(b, b^{x_4})))$$

And, with two more inferences by *marking right*, we get

$$10) \quad \rightarrow g(f(b^{x_1}, b^{x_2})) \simeq g(g(f(b, b^{x_4})))$$
$$11) \quad \rightarrow g(f(b^{x_1}, b^{x_2})) \simeq g(g(f(b^{x_3}, b^{x_4})))$$

It is worth noting that clause 1 follows from the smaller clause 7, clauses 9 and 10 follow from the smaller clause 11, and clause 8 follows from the smaller clauses 7 and 11. Hence, using *redundancy of clauses* (c.f. Sect. 6), clauses 1, 8, 9 and 10 could be deleted.

Now recall that $\succ$ fulfills the subterm property. Therefore we have that $\mathit{Forget}(g(g(f(b^{x_3}, b^{x_4})))) \succ \mathit{Forget}(g(f(b^{x_1}, b^{x_2})))$, and hence an inference by *paramodulation left* with 11 into 6 gives us

$$12) \quad g(f(b^{x_1}, b^{x_2})) \simeq g(f(b^{x_1}, b^{x_2})) \rightarrow$$

And finally, the empty clause is obtained with an inference by *equality resolution*.

Now let us show that, if the blockings were not introduced (in other words, if inferences by *paramodulation right* with 1 instead of 7 were allowed), then refutation completeness would be lost. If an inference by *paramodulation right* with 1 into 5 was possible, then it would give either

$$\rightarrow g(f(b, b)) \simeq g(g(f(b, a)))$$

or

$$\rightarrow g(f(a, b)) \simeq g(g(f(b, b))),$$

which could be simplified back into 5 respectively with 2 and 3, which are included in the reduction ordering. Moreover, inferences by *paramodulation right* with 1 into 2 and 3 give us 4. And inferences by *paramodulation right* between 2, 3 and 4 give us already existing clauses or clauses like

$$\rightarrow f(a, b) \simeq f(b, a),$$

which follow from the smaller (w.r.t. the subterm relation) clause 1. Finally, since $g(f(a,b)) \succ g(g(f(b,a)))$, an inference by *paramodulation left* with 5 into 6 gives us

$$7) \quad f(a,b) \simeq g(g(f(b,a))) \rightarrow$$

but, since $f(a,b) \simeq g(g(f(b,a)))$ does not follow from the positive equations, further inferences on 7 will not lead to the empty clause. Therefore the set S can be saturated without the empty clause being derived. $\square$

As usual, useful methods like *subsumption* of clauses can be covered, for instance, by more general notions of redundancy for clauses like *non-strict redundancy*, together with the notion of *ground fairness* [NR01] (see also [BG94] for an alternative way of capturing subsumption).

8 Knuth-Bendix Completion

In this section, E denotes a set of equations and $\succ_r$ denotes a (possibly non-totalizable) reduction ordering on $T(\mathcal{F}, \mathcal{X})$. Then a *convergent TRS R for E and $\succ_r$* is a convergent TRS, logically equivalent to E, and such that $l \succ_r r$ for all its rules $l \rightarrow r$. The problem we deal with is finding a convergent TRS for the given E and $\succ_r$ whenever it exists, and find it in finite time if it is finite. Note that being a reduction ordering is the minimal possible requirement on $\succ_r$, since a TRS terminates if, and only if, it is contained in a reduction ordering.

It is not difficult to devise a procedure of theoretical nature for doing this by enumerating all equational consequences of E. However, finding practically useful procedures remained as an open problem for a long time. Devie showed that for left- and right *linear* E (i.e. no variable occurs more than once in a side of an equation) standard Knuth-Bendix completion finds R [Dev90]. But it was not until in [BGNR99,BGNR03] that a procedure for the general problem, not relying on the enumeration of all equational consequences, was presented.

Here we improve the method presented in [BGNR03] in that we make it compatible with powerful redundancy notions, like simplification by rewriting. The proof is a straightforward adaptation of the one in [BGNR03] since, in the limit, an interreduced convergent TRS is generated and, hence, the same techniques can be applied.

For every convergent TRS, its unique *canonical* version can be obtained by *interreducing* it by (i) normalizing all right-hand sides and (ii) removing rules whose left-hand sides are reducible by other rules [DMT88]. The following lemma strengthens the uniqueness result of [DMT88] to TRS included in a west-ordering, but only for the ground case:

Lemma 15. *[BGNR03] Let $\succ$ be a west-ordering, and let R_1 and R_2 be interreduced TRS over $T(\mathcal{F})$, both included in $\succ$, and such that $R_1^* = R_2^*$. Then $R_1 = R_2$.*

In the following, R denotes the canonical TRS for E and $\succ_r$ (we assume that R exists), R_g denotes the canonical TRS for $gnd(R)$, the set of all ground instances of rules of R, and $\succ$ denotes a west ordering including $\succ_r$. We will use the same inference system as in Definition 4, but where the *paramodulation* inference is relaxed by changing condition 4 as follows:

4. for some ground substitution θ, we have $\mathit{Forget}((l \cdot \delta)\sigma\theta) \succ \mathit{Forget}((r \cdot \delta)\sigma\theta)$.

With this change, now the ordered paramodulation rule is also applied on top of the small sides. This is necessary in order to obtain an interreduced ground TRS for the model. We denote the resulting inference system by $\mathcal{K}$.

By R_E, R_T and R_{ET} we denote the TRSs generated from a set of marked equations E and a set of terms T as in Definition 7, but where in case 1d total irreducibility is required instead of just irreducibility at non-topmost positions. Then it follows that $\mathit{Forget}(R_E)$ is interreduced and included in $\succ$.

Lemma 16. *For all rules $l \to r$ in $\mathit{Forget}(R_E)$ we have that*

1. *r is irreducible by $\mathit{Forget}(R_E)$, and*
2. *l is irreducible by $\mathit{Forget}(R_E) \setminus \{l \to r\}$.*

Moreover, since an equation $s \simeq t$ can be identified with a clause $\to s \simeq t$, the dynamic redundancy notions and lemmas for clauses of Sect. 6 hold for equations as well.

Theorem 3. *Let $\langle E_0, T_0\rangle, \langle E_1, T_1\rangle, \ldots$ be a fair derivation with respect to $\mathcal{K}$, where E_0 is the given set of equations E, and let R_{E_∞} denote the TRS R_E obtained taking $E = E_\infty$ and $T = T_\infty$. Then $\mathit{Forget}(R_{E_\infty})^* \models \mathit{Forget}(\mathit{gnd}(E_\infty))$.*

Proof. By Lemma 14 we have that $\langle E_\infty, T_\infty\rangle$ is saturated with respect to $\mathcal{K}$. Then we proceed analogously to the proof of Theorem 1, with the only difference that inferences at topmost positions of small sides of equations also are considered. $\square$

Lemma 17. *Let $\langle E_0, T_0\rangle, \langle E_1, T_1\rangle, \ldots$ be a fair derivation with respect to $\mathcal{K}$, where E_0 is the given set of equations E. Then $R_g = \mathit{Forget}(R_{E_\infty})$.*

Proof. By Theorem 3, we have $\mathit{Forget}(R_{E_\infty})^* \models \mathit{Forget}(\mathit{gnd}(E_\infty))$, and hence $\mathit{Forget}(R_{E_\infty})^* \models \mathit{Forget}(E_\infty)$. Then, by Lemma 13, we have $\mathit{Forget}(R_{E_\infty})^* \models E$. Therefore $E^* = \mathit{Forget}(R_{E_\infty})^*$, since all rules in $\mathit{Forget}(R_{E_\infty})$ are logical consequences of E. Furthermore, we clearly have $E^* = R^* = R_g^*$, and hence $R_g^* = \mathit{Forget}(R_{E_\infty})^*$. We also have $R_g \subseteq \succ_r \subseteq \succ$, and $\mathit{Forget}(R_{E_\infty}) \subseteq \succ$. Therefore, since both are interreduced we conclude by Lemma 15 that $R_g = \mathit{Forget}(R_{E_\infty})$. $\square$

Then, by the same arguments as in [BGNR03] we get the following result.

Theorem 4. *Let $\langle E_0, T_0\rangle, \langle E_1, T_1\rangle, \ldots$ be a fair derivation with respect to $\mathcal{K}$, where E_0 is the given set of equations E. Then $\mathit{Forget}(E_\infty) \supseteq R$.*

Proof. W.l.o.g. assume there are sufficiently many new constants in $\mathcal{F}$ that do not occur in E or R. Let $l \to r$ be an arbitrary rule in R. We prove that $l \simeq r \in \mathit{Forget}(E_\infty)$. Let σ be the ground substitution replacing each variable with a distinct new constant. Then $l\sigma \to r\sigma$ is in $gnd(R)$ and also in R_g: since R is interreduced, r is irreducible by R (and hence $r\sigma$ by $gnd(R)$), and l is irreducible by $R \setminus \{l \to r\}$ (and hence $l\sigma$ by $gnd(R) \setminus \{l\sigma \to r\sigma\}$). If $l\sigma \to r\sigma$ is in R_g and by the previous lemma $R_g = \mathit{Forget}(R_{E_\infty})$, then $l\sigma \to r\sigma$ is in $\mathit{Forget}(R_{E_\infty})$. Since the new constants do not occur in E, if $l\sigma \to r\sigma \in \mathit{Forget}(R_{E_\infty})$, then some $l' \simeq r'$ is in $\mathit{Forget}(E_\infty)$, and $l\sigma \equiv l'\theta\sigma$ and $r\sigma \equiv r'\theta\sigma$ for some θ. We conclude by showing that θ is the identity substitution: there is no equation $l' \simeq r'$ in $\mathit{Forget}(E_\infty)$ that strictly subsumes a rule in R; otherwise, the rewrite proof by R of $l' \simeq r'$ would apply some rule different from $l \to r$, which would then also reduce $l \to r$, contradicting $l \to r \in R$. □

Therefore, if R is finite, it will be contained in $\mathit{Forget}(E_j)$ for some E_j of the derivation.

9 General clauses

The extension of the presented calculus to general first order clauses is not straightforward since, roughly, the orderings must be carefully extended or the inference system be adapted so that, in the completeness proof, the conclusion of an inference be always smaller than the maximal premise (even in the case of a positive strategy, paramodulation with non-unit clauses is clearly necessary and, hence, some equations can be added to the conclusion of an inference, which moreover need be compared with the rightmost premise). The approach we present here consists in using almost the same ordering $\succ_i^c$ as in the Horn case, and roughly marking all terms which are "introduced" in the conclusion and are not smaller w.r.t. $\succ_{red}$ than some term in the rightmost premise. This is in some sense the natural extension, to the case of general clauses, of the strategy used under the Horn case, but, when applying paramodulation with non-unit clauses, it tends to generate many marks. At present we do not know whether these additional marks (w.r.t. the ones we already added in the Horn case) are necessary to preserve completeness.

Here we will deal, as in the Horn case, with a set of clauses S and a set of terms T, which contains the set of terms that are going to be marked. We will analogously have a *mark detection* rule which adds terms to the set T but, additionally, some terms will be added to the set T when performing inferences. An alternative to this approach could be to have additional mark detection rules (and non let inference rules adding terms to T), but this would not allow us to keep the set T as reduced as possible.[7]

We extend the definition of $\mathit{Marks}(S)$ for a multiset of marked terms S given in Section 4.1, to multisets of multisets of marked terms as follows: if

[7] Technically, although now the inference rules can also add terms to the set T, a mark detection rule is still necessary for the lifting argument.

S is a multiset $\{S_1,\ldots,S_n\}$ of multisets of marked terms, then $\mathit{Marks}(S) = \bigcup_{i\in 1\ldots n}\mathit{Marks}(S_i)$.

As for the Horn case, in the following inference system we consider that in each clause with a non-empty antecedent one of its negative equations, the one that is written underlined, has been selected. We also assume that all terms in the premises of an inference share a unique substitution.

Definition 10. *Let S be a set of general first order clauses and T be a set of terms, which satisfy the following condition: for every marked subterm $(s\cdot\gamma)^x$ occurring in S, we have that $\mathit{Forget}(s\cdot\gamma)$ is an instance of a term in T. Our inference system $\mathcal{I}$ consists of the following six inference rules and a mark detection rule:*

Paramodulation right:

$$\frac{\rightarrow l\cdot\gamma\simeq r\cdot\gamma, \Delta_1 \qquad \rightarrow s\cdot\gamma\simeq t\cdot\gamma, \Delta_2}{(\rightarrow s'\cdot\gamma'\simeq t\cdot\gamma, \Delta_1', \Delta_2)\sigma} \qquad \textit{if}$$

1. *$\sigma = mgu(l\cdot\gamma, s\cdot\gamma|_p)$ for some position p,*
2. *$\mathit{Forget}(s\cdot\gamma|_p)$ is not a variable,*
3. *$\mathit{Forget}(s'\cdot\gamma')\equiv\mathit{Forget}(s\cdot\gamma[r\cdot\gamma]_p)$, and $\mathit{Forget}(\Delta_1')$ is the same as $\mathit{Forget}(\Delta_1)$,*
4. *for some ground substitution θ, we have $\mathit{Forget}((l\cdot\gamma)\sigma\theta)\succ\mathit{Forget}((r\cdot\gamma)\sigma\theta)$ and $\mathit{Forget}((l\cdot\gamma\simeq r\cdot\gamma)\sigma\theta)\succ^{mul}\mathit{Forget}(e\sigma\theta)$ for all equations e in Δ_1 and, if $p=\lambda$, then $\mathit{Forget}((s\cdot\gamma)\sigma\theta)\succ\mathit{Forget}((t\cdot\gamma)\sigma\theta)$ and $\mathit{Forget}((s\cdot\gamma\simeq t\cdot\gamma)\sigma\theta)\succ^{mul}\mathit{Forget}(e\sigma\theta)$ for all equations e in Δ_2,*
5. *$(s\cdot\gamma)\sigma\succ_{red}(s'\cdot\gamma')\sigma$, or $s\doteq s'$ and $s|_q$ is a variable for some prefix of q of p, and*
6. *for every equation e in Δ_1', $\{s\cdot\gamma, t\cdot\gamma\}\sigma\succ_{red}^{mul} e\sigma$ or both sides of e are marked on top,*

where, moreover, the set of terms

$$\{\mathit{Forget}(u\cdot\rho)\sigma \mid u\cdot\rho\in\mathit{Marks}(\{s'\cdot\gamma'\}\cup\Delta_1') \\ \text{and } \mathit{Forget}(u\cdot\rho) \text{ is not an instance of a term in } T\}$$

is added to T whenever the conclusion is added to S.

Observe that, for some parts of the conclusion, it is not precisely specified what positions should be marked. For instance, one strategy satisfying condition 5 could be to take $s\cdot\gamma[r\cdot\gamma]_p$ as $s'\cdot\gamma'$ whenever $(s\cdot\gamma)\sigma\succ_{red}(s\cdot\gamma[r\cdot\gamma]_p)\sigma$ or p is below a marked position of $s\cdot\gamma$, and take $s\cdot\gamma[(r\cdot\gamma)^x]_p$, where x is a fresh variable, otherwise (which makes $(s\cdot\gamma)\sigma\succ_{red}(s'\cdot\gamma')\sigma$, since $s\gtrdot s'$); this is the strategy we took in the equational and Horn cases, for clarity reasons. One simple strategy satisfying condition 6 could be to take Δ_1' as Δ_1, but where every side of an equation which is not smaller w.r.t. $\succ_{red}$, under the substitution σ, than $(s\cdot\gamma)\sigma$ or $(t\cdot\gamma)\sigma$, has been marked on top. However, more sophisticated strategies can be considered, including the possibility of forgetting *some marks in the conclusion:*

as an example, if the right premise has no marks and $(s \simeq t)\sigma \succ_r^e$ *Forget*$(e)\sigma$ *for every equation* e *in* Δ_1*, then we can take Forget*(Δ_1) *as* Δ_1'*.*

Paramodulation left:

$$\frac{\rightarrow l \cdot \gamma \simeq r \cdot \gamma, \Delta_1 \qquad \Gamma, \underline{s \cdot \gamma \simeq t \cdot \gamma} \rightarrow \Delta_2}{(\Gamma, s' \cdot \gamma' \simeq t \cdot \gamma \rightarrow \Delta_1', \Delta_2)\sigma} \qquad \text{if}$$

1. $\sigma = mgu(l \cdot \gamma, s \cdot \gamma|_p)$ *for some position* p*,*
2. *Forget*$(s \cdot \gamma|_p)$ *is not a variable,*
3. *Forget*$(s' \cdot \gamma') \equiv$ *Forget*$(s \cdot \gamma[r \cdot \gamma]_p)$*, and Forget*$(\Delta_1')$ *is the same as Forget*(Δ_1)*,*
4. *for some ground substitution* θ*, we have Forget*$((l \cdot \gamma)\sigma\theta) \succ$ *Forget*$((r \cdot \gamma)\sigma\theta)$ *and Forget*$((l \cdot \gamma \simeq r \cdot \gamma)\sigma\theta) \succ^{mul}$ *Forget*$(e\sigma\theta)$ *for all equations* e *in* Δ_1 *and, if* $p = \lambda$*, then Forget*$((s \cdot \gamma)\sigma\theta) \succ$ *Forget*$((t \cdot \gamma)\sigma\theta)$*.*
5. $(s \cdot \gamma)\sigma \succ_{red} (s' \cdot \gamma')\sigma$*, or* $s \doteq s'$ *and* $s|_q$ *is a variable for some prefix of* q *of* p*, and*
6. *for every equation* e *in* Δ_1'*,* $\{s \cdot \gamma, s \cdot \gamma, t \cdot \gamma, t \cdot \gamma\}\sigma \succ_{red}^{mul} e\sigma$ *or both sides of* e *are marked on top,*

where, moreover, the set of terms

$$\{\mathit{Forget}(u \cdot \rho)\sigma \mid u \cdot \rho \in \mathit{Marks}(\{s' \cdot \gamma'\} \cup \Delta_1') \\ \text{and } \mathit{Forget}(u \cdot \rho) \text{ is not an instance of a term in } T\}$$

is added to T *whenever the conclusion is added to* S*.*

Marking right:

$$\frac{\rightarrow s \cdot \gamma[u \cdot \delta]_p \simeq t \cdot \gamma, \Delta}{(\rightarrow s \cdot \gamma[(u \cdot \delta)^x]_p \simeq t \cdot \gamma, \Delta)\sigma} \qquad \text{if}$$

1. $\sigma = mgu($*Forget*$(u \cdot \delta), u')$ *for some term* u' *in* T*,*
2. u *is not a variable, or* u' *is a variable and* $p = \lambda$*, and*
3. x *is a fresh variable.*

Marking left:

$$\frac{\Gamma, \underline{s \cdot \gamma[u \cdot \delta]_p \simeq t \cdot \gamma} \rightarrow \Delta}{(\Gamma, s \cdot \gamma[(u \cdot \delta)^x]_p \simeq t \cdot \gamma \rightarrow \Delta)\sigma} \qquad \text{if}$$

1. $\sigma = mgu($*Forget*$(u \cdot \delta), u')$ *for some term* u' *in* T*,*
2. u *is not a variable,*
3. x *is a fresh variable, and*
4. *if* $p = \lambda$ *then, for some substitution* θ*, we have that Forget*$((s \cdot \gamma)\sigma\theta) \succ$ *Forget*$((t \cdot \gamma)\sigma\theta)$*.*

Equality resolution:

$$\frac{\Gamma, \underline{s\cdot\gamma \simeq t\cdot\gamma} \rightarrow \Delta}{(\Gamma \rightarrow \Delta)\sigma} \qquad \textit{if} \quad \sigma = mgu(s\cdot\gamma, t\cdot\gamma).$$

Equality factoring:

$$\frac{\rightarrow l\cdot\gamma \simeq r\cdot\gamma, s\cdot\gamma \simeq t\cdot\gamma, \Delta}{(r'\cdot\gamma' \simeq t'\cdot\gamma' \rightarrow l\cdot\gamma \simeq r\cdot\gamma, \Delta)\sigma} \qquad \textit{if}$$

1. $\sigma = mgu(l\cdot\gamma, s\cdot\gamma)$,
2. $\mathit{Forget}(r'\cdot\gamma') \equiv \mathit{Forget}(r\cdot\gamma)$ *and* $\mathit{Forget}(t'\cdot\gamma') \equiv \mathit{Forget}(t\cdot\gamma)$,
3. *for some ground substitution* θ, *we have* $\mathit{Forget}((l\cdot\gamma)\sigma\theta) \succ \mathit{Forget}((r\cdot\gamma)\sigma\theta)$ *and* $\mathit{Forget}((l\cdot\gamma \simeq r\cdot\gamma)\sigma\theta) \succeq^{mul} \mathit{Forget}(e\sigma\theta)$ *for all equations* e *in* $\{s\cdot\gamma \simeq t\cdot\gamma\} \cup \Delta$, *and*
4. $(s\cdot\gamma)\sigma \succ_{red} (r'\cdot\gamma')\sigma$ *or* $r'\cdot\gamma'$ *is marked on top, and* $(s\cdot\gamma)\sigma \succ_{red} (t'\cdot\gamma')\sigma$ *or* $t'\cdot\gamma'$ *is marked on top,*

where, moreover, the set of terms

$$\{\mathit{Forget}(u\cdot\rho)\sigma \mid u\cdot\rho \in \mathit{Marks}(\{r'\cdot\gamma', t'\cdot\gamma'\}) \\ \textit{and } \mathit{Forget}(u\cdot\rho) \textit{ is not an instance of a term in } T\}$$

is added to T *whenever the conclusion is added to* S.

Mark detection:

$$T \Longrightarrow T \cup \{\mathit{Forget}(r\cdot\gamma)\} \qquad \textit{if}$$

1. $\rightarrow l\cdot\gamma \simeq r\cdot\gamma, \Delta \in S$,
2. $l \not\succ_r r$ *and* $r\cdot\gamma$ *is not marked on top, and*
3. *for some ground substitution* θ, *we have* $\mathit{Forget}((l\cdot\gamma)\theta) \succ \mathit{Forget}((r\cdot\gamma)\theta)$ *and* $\mathit{Forget}((l\cdot\gamma \simeq r\cdot\gamma)\theta) \succ^{mul} \mathit{Forget}(e\theta)$ *for all equations* e *in* Δ.

Observe that the previous inference system preserves the invariant that for every marked subterm $(s\cdot\gamma)^x$ occurring in S, $\mathit{Forget}(s\cdot\gamma)$ is an instance of a term in T.

The following is a toy example to illustrate how the previous inference system works.

Example 5. Let $\succ_r$ be a reduction ordering including

$$g(a) \succ_r g(f(a)) \succ_r h(f(a)) \succ_r h(a), \\ h(a) \succ_r f(a), \text{ and } h(a) \succ_r a.$$

Observe that, on the one hand, $f(a)$ and a must be incomparable in $\succ_r$ and, on the other hand, we must have $f(a) \succ a$ in any west ordering $\succ$ including $\succ_r$.

Let S denote the following inconsistent set of general first order clauses,

$$\begin{array}{ll} 1) & \rightarrow g(f(a)) \simeq h(a), g(a) \simeq h(f(a)) \\ 2) & g(x) \simeq h(y) \rightarrow x \simeq y \\ 3) & g(x) \simeq h(x) \rightarrow \end{array}$$

and let $T = \emptyset$.

We illustrate how the empty clause can be derived from the pair $\langle S, T\rangle$ with the inference system $\mathcal{I}$. In order to ease the reading, we sometimes underline the terms that are unified in an inference.

First of all, an inference by *paramodulation left* with 1 into 2, that is with $\rightarrow g(f(a)) \simeq h(a), \underline{g(a)} \simeq h(f(a))$ into $\underline{g(x)} \simeq h(y) \rightarrow x \simeq y$ is possible, giving

$$4) \quad h(f(a)) \simeq h(y) \rightarrow a \simeq y, g(f(a)) \simeq h(a)$$

Observe that this is a valid inference, where $\sigma = \{x \mapsto a\}$, since $g(a)$ is strictly maximal w.r.t. $\succ$ in the left premise (where $\succ$ is the west ordering including $\succ_r$), and $g(x)\sigma\theta \succ h(y)\sigma\theta$ for some ground substitution θ (take, e.g., $\theta = \{y \mapsto a\}$). Moreover, no terms needs to be marked in the conclusion, since $g(a) \succ_r h(f(a))$ (and hence $g(a) \succ_{red} h(f(a))$), and $\{g(a), h(y)\} \succ_r^{mul} \{g(f(a)), h(a)\}$ (and hence $\{g(a), h(y)\} \succ_{red}^{mul} \{g(f(a)), h(a)\}$).

Now, an inference by *equality resolution* on 4 is possible, giving

$$5) \quad \rightarrow a \simeq f(a), g(f(a)) \simeq h(a)$$

Then an inference by *paramodulation left* with 5 into 2, that is with $\rightarrow a \simeq f(a), \underline{g(f(a))} \simeq h(a)$ into $\underline{g(x)} \simeq h(y) \rightarrow x \simeq y$ is possible, giving

$$6) \quad h(a) \simeq h(y) \rightarrow f(a) \simeq y, a \simeq f(a)$$

Again, no mark needs to be added to the conclusion, since $g(f(a)) \succ_r h(a)$ and $\{g(f(a)), h(y)\} \succ_r^{mul} \{a, f(a)\}$.

Now, an inference by *equality resolution* on 6 gives us

$$7) \quad f(a) \simeq a, a \simeq f(a)$$

And with an inference by *equality factoring* on 7 we obtain

$$8) \quad a \simeq a \rightarrow f(a) \simeq a$$

Note that this is a valid inference, since $f(a) \succ a$ and $\{f(a), a\} \succeq^{mul} \{a, f(a)\}$. Moreover, no mark is added to the conclusion since, although $f(a) \not\succ_r a$, we have that $f(a) \rhd a$ and hence $f(a) \succ_{red} a$.

Now with an inference by *equality resolution* on 8 we get

$$9) \quad \rightarrow f(a) \simeq a$$

Then there is an inference by *paramodulation right* with 9 into 1, that is with $\rightarrow \underline{f(a)} \simeq a$ into $\rightarrow g(\underline{f(a)}) \simeq h(a), g(a) \simeq h(f(a))$, giving

$$10) \quad \rightarrow g(a^{x_1}) \simeq h(a), g(a) \simeq h(f(a))$$

Here a mark is necessary at the inference position, since neither $g(f(a)) \succ_r g(a)$ nor $g(f(a)) \rhd g(a)$ (in fact, we have $g(a) \succ_r g(f(a))$); by introducing the mark, we obtain $g(f(a)) \succ_{red} g(a^{x_1})$ (since $g(f(a)) \rhd g(x_1)$). Recall that after this, the term a is added to the set T of terms that are going to be marked.

A new inference by *paramodulation right* with 9 into 10, that is with $\rightarrow \underline{f(a)} \simeq a$ into $\rightarrow g(a^{x_1}) \simeq h(a), g(a) \simeq h(\underline{f(a)})$, gives us

$$11) \quad \rightarrow g(a^{x_1}) \simeq h(a), g(a) \simeq h(a)$$

In this case a mark is not necessary at the inference position, since $h(f(a)) \succ_r h(a)$.

Now, since $a \in T$, an inference by *marking right* on 11 is possible, and we obtain

$$12) \quad \rightarrow g(a^{x_1}) \simeq h(a), g(a^{x_2}) \simeq h(a)$$

After this, $g(a^{x_1})$ and $g(a^{x_2})$ unify, and hence an inference by *equality factoring* on 12 gives us

$$13) \quad h(a) \simeq h(a) \rightarrow g(a^{x_1}) \simeq h(a)$$

Then, with an inference by *equality resolution* on 13 we get

$$14) \quad \rightarrow g(a^{x_1}) \simeq h(a)$$

And an inference by *paramodulation left* with 14 into 3, that is with $\rightarrow \underline{g(a^{x_1})} \simeq h(a)$ into $g(x) \simeq h(x) \rightarrow$, gives us

$$15) \quad h(a)^{x_2} \simeq h(a^{x_1}) \rightarrow$$

Observe that a mark is introduced at the inference position in the conclusion. This is because we are assuming that $g(x_1) \not\succ_r h(a)$ for all x_1. However, in the case that, for example, we had $g(x) \succ_r h(x)$ for all x, $h(a^{x_1}) \simeq h(a^{x_1}) \rightarrow$ would also be a valid conclusion. In the present case, since we mark the whole term, $h(a)$ is added to T, and hence after this inference $T = \{a, h(a)\}$.

Then, with an inference by *marking left* on 15 we get

$$16) \quad h(a^{x_1})^{x_2} \simeq h(a^{x_1}) \rightarrow$$

And with another inference by *marking left*, we get

$$17) \quad h(a^{x_1})^{x_2} \simeq h(a^{x_1})^{x_2} \rightarrow$$

Finally, with an inference by *equality resolution* on 17 the empty clause is obtained. $\square$

9.1 Redundancy notions

Here, as for the case of Horn clauses, the notions of redundancy and saturation are obtained from the ones given in Section 4.1, by using clauses instead of equations and the inference system $\mathcal{I}$ instead of $\mathcal{E}$, and lifting the orderings adequately. Moreover, since the inference rules are non-deterministic, an inference is defined to be redundant also when some of the possible conclusions is included in the set of clauses S. Since now inference rules can also add terms to the set T, we refer to the inferences as using a set of terms T as well as some clauses as premises. However, only for the case of *marking* inferences the set T determines whether the inference can be performed.

Definition 11. *Let π be an inference by $\mathcal{I}$, using a set of terms T, with premises $C_1, \ldots C_n$ and conclusion D. Then a* ground instance $\pi\theta$ of the inference π *is an inference by $\mathcal{I}$, using T, with premises $C_1\theta, \ldots C_n\theta$ and conclusion $D\theta$ for some ground substitution θ.*

Let G be a set of ground marked clauses. A ground inference by $\mathcal{I}$, using some set of terms T, with premises $C_1, \ldots C_n$ and conclusion D is redundant in G *if*

$$Forget(G^{\prec^c_{red} C_n}) \cup Forget(G^{\prec^c_{red} D}) \models Forget(D)$$

or there is a clause D' in G such that D' is a valid conclusion for the same premises $C_1, \ldots, C_n$ and the same set of terms T.

Let S be a set of marked clauses. An inference by $\mathcal{I}$, using some set of terms T, with premises $C_1, \ldots C_n$ and conclusion D is redundant in S *if, for all sets of terms M and ground substitutions θ with marks in M, $\pi\theta$ is redundant in $gnd_M(S)$.*

Let T be a set of terms. A mark detection rule $T \Longrightarrow T \cup \{t\}$ is redundant in T if t is an instance of a term in T.

As usual, from redundancy of inferences, saturation with respect the inference system can be defined.

Definition 12. *Let S be a set of clauses and T be a set of terms. The pair $\langle S, T\rangle$ is* saturated *with respect to $\mathcal{I}$ if every inference by $\mathcal{I}$, using T and with premises in S, is redundant in S, and every mark detection rule is redundant in T.*

We take the same notion of redundancy for clauses as in Section 6, and also same notion of *derivation*, but with an additional possibility for removing terms from the set T, based on the fact that now it is likely that some marks can disappear from S when deleting non-unit clauses (e.g., when a clause like $\rightarrow l \cdot \gamma \simeq r \cdot \gamma, \Delta$ is deleted in the presence of $\rightarrow \Delta$ and the marks in $l \cdot \gamma \simeq r \cdot \gamma$ do not occur in any other clause). Then case 2b becomes:

2b. $T_{i+1} = T_i \setminus \{t\}$ where t is an instance of a term in $T_i \setminus \{t\}$, or t has not been obtained by a mark detection rule and there is no marked subterm of the form $(s \cdot \gamma)^x$ occurring in S such that $Forget(s \cdot \gamma)$ is an instance of t.

From the notion of redundancy for clauses, the following lemma holds as well.

Lemma 18. *Let $\langle S_0, T_0\rangle, \langle S_1, T_1\rangle, \ldots$ be a derivation and let C be a clause in $(\cup_i S_i) \setminus S_\infty$. Then C is redundant in S_∞.*

The notion of fairness is the same as in Section 6. Then Lemma 14, applied to $\mathcal{I}$, holds as well:

Lemma 19. *If $\langle S_0, T_0\rangle, \langle S_1, T_1\rangle, \ldots$ is a fair derivation with respect to $\mathcal{I}$, then $\langle S_\infty, T_\infty\rangle$ is saturated with respect to $\mathcal{I}$.*

9.2 Model generation

Analogously to Section 4.2, we denote by T_{S} the set of positive unit clauses $\rightarrow t \simeq \bot$ for all t in T, and we denote by T_{G} the set of mark free ground instances of clauses in T_{S}. The model will be defined inductively on the clauses in T_{G} and $gnd_T(S)$ with respect to $\succ_{mod}$, a total well-founded ordering including: $C \succ_{mod} D$ if

1. $\mathit{Forget}(C) \succ^c \mathit{Forget}(D)$.
2. $\mathit{Forget}(C) \equiv^c \mathit{Forget}(D)$ and $C \succ^c_m D$, which means that D has marks at higher positions than C.

Now we define a rewrite system R_S from $gnd_T(S)$, which requires to define at the same time a rewrite system R_T from T_{G}.

Definition 13. *The rewrite system R_{ST} is the union of R_S and R_T. Let C be a ground (marked) clause in $gnd_T(S) \cup T_{\mathrm{G}}$.*

1. *If C is an instance $\rightarrow (s\cdot\gamma \simeq t\cdot\gamma)\theta, \Delta\theta$ in $gnd_T(S)$ of some clause $\rightarrow s\cdot\gamma \simeq t\cdot\gamma, \Delta$ in S, then it generates the rule $(s\cdot\gamma)\theta \rightarrow (t\cdot\gamma)\theta$ in R_S if*
   (a) *$\mathit{Forget}(R_S^C)^* \not\models \mathit{Forget}(C)$,*
   (b) *$s \succ_r t$ or $t\cdot\gamma$ is marked on top,*
   (c) *$\mathit{Forget}((s\cdot\gamma)\theta) \succ \mathit{Forget}((t\cdot\gamma)\theta)$ and $\mathit{Forget}((s\cdot\gamma \simeq t\cdot\gamma)\theta) \succ^{mul} \mathit{Forget}(e\theta)$ for all equations e in Δ,*
   (d) *$(s\cdot\gamma)\theta$ is irreducible by $R_S^C \cup R_T^C$, and*
   (e) *$(t\cdot\gamma)\theta$ and $\Delta\theta$ are irreducible by $R_S^C \cup R_T^C$ at non-topmost positions.*
   (f) *$\mathit{Forget}(R_S^C)^* \models \mathit{Forget}((r\cdot\gamma \simeq t\cdot\gamma)\theta)$ for no equation $l\cdot\gamma \simeq t\cdot\gamma$ in Δ,*
2. *If C is an instance $\rightarrow t\theta \simeq \bot$ in T_{G} of a clause $\rightarrow t \simeq \bot$ in T_{S}, then it generates the rule $u\cdot\delta \rightarrow (u\cdot\delta)^x$ in R_T if $u\cdot\delta$ is the normal form of $t\theta$ w.r.t. R_T^C and $u\cdot\delta$ is irreducible by R_S^C,*

where R_S^C and R_T^C denote the set of rules in R_S and R_T respectively, generated by clauses D such that $C \succ_{mod} D$.

By construction, as in Section 4.2, we obtain the following analogous results:

Lemma 20. *Let $l\cdot\delta \rightarrow r\cdot\delta$ be a rule in R_T. Then $r\cdot\delta \equiv (l\cdot\delta)^x$, $l\cdot\delta$ is not marked on top, and the rule has been generated by a clause of the form $\rightarrow t\theta \simeq \bot$, where $t \in T$, $t\theta \in T(\mathcal{F})$, and $\mathit{Forget}(l\cdot\delta) \equiv t\theta$.*

Lemma 21. *Let $s \cdot \gamma$ be a marked subterm occurring in R_{ST}. If $s \cdot \gamma$ is marked on top, then Forget$(s \cdot \gamma)$ is an instance of a term in T.*

Lemma 22. *Let $s \cdot \gamma$ be a marked subterm occurring, not as right-hand side, in some rule of R_S. Then $s \cdot \gamma$ is marked on top if, and only if, Forget$(s \cdot \gamma)$ is an instance of a term in T.*

Lemma 23. *For all rules $l \to r$ in Forget(R_S) we have that*

1. *r is irreducible by Forget(R_S) at non-topmost positions, and*
2. *l is irreducible by Forget$(R_S) \setminus \{l \to r\}$.*

Lemma 24. *Forget(R_S) is convergent.*

The following lemma holds analogously as in [BGNR03]:

Lemma 25. *If a clause $\to l \cdot \delta \simeq r \cdot \delta, \Delta$ in $gnd_T(S)$ generates the rule $l \cdot \delta \to r \cdot \delta$ in R_S, then Forget(Δ) is irreducible by Forget(R_S) at non-topmost positions and Forget$(R_S)^* \not\models$ Forget(Δ).*

9.3 Completeness

We now introduce the well-founded ordering $\succ_j$ that will be used in the proof of Theorem 5. It extends $\succ_i$ (see Definition 3) with an additional case:

Definition 14. *Let $s \cdot \gamma$ and $t \cdot \delta$ be two marked terms, and let R denote Forget(R_S). Then $s \cdot \gamma \succ_j t \cdot \delta$ iff (i) $s \rhd \cup \succ_r t$ or (ii) $s \doteq t$ and Ran$(\gamma) \succ_m^{mul}$ Ran(δ) or (iii) $s \doteq t$ and Forget(Ran(γ)) $\xrightarrow{+}_R$ Forget(Ran(δ)) or (iv) $s \doteq t$, all terms in Forget(Ran(δ)) are irreducible by R at non-topmost positions and Forget(Ran(γ)) $\succ^{mul}$ Forget(Ran(δ)).*

In what follows, we will consider $\succ_j$ as the transitive closure of the ordering defined above (composed with the congruence $\equiv$ on marked terms).

Property 1. Let R denote *Forget*(R_S). Then for all ground terms s and t such that s is irreducible by R at non-topmost positions and $s \to_R^+ t$, we have

1. $s \succ t$ and
2. t is irreducible by R_S at non-topmost positions.

Proof. If $s \to_R s_1 \to_R \cdots \to_R s_n \to_R t$ then, since s is irreducible at non-topmost positions and, by Lemma 23, all right-hand sides of rules in R are irreducible at non-topmost positions, all steps in the sequence are at topmost position, which implies on the one hand that t is irreducible at non-topmost positions and, on the other hand, by definition of R_S, that $s \succ s_1 \succ \cdots \succ s_n \succ t$.

Lemma 26. *$\succ_j$ is well-founded.*

Proof. Since the relation $(\rhd \cup \succ_r) \circ \doteq$ is well-founded (see [DJ90]), in a sequence $s_1 \cdot \gamma_1 \succ_j s_2 \cdot \gamma_2 \cdots$ there can only be finitely many steps by case (i). Hence, from some point on, there are only steps by case (ii), (iii) or (iv). Now observe that, if $s_k \cdot \gamma_k \succ_j s_m \cdot \gamma_m \succ_j s_n \cdot \gamma_n$, where the first step is by case (ii) and the second step is by case (iii) or (iv), then $s_k \cdot \gamma_k \succ_j s_n \cdot \gamma_n$ by case (iii) or (iv), since $\mathit{Forget}(s_k \cdot \gamma_k) \equiv \mathit{Forget}(s_m \cdot \gamma_m)$ by definition of $\succ_m$. But there must be only finitely many steps by (iii) or (iv), since R is terminating, $\succ$ is well-founded and, by Property 1, after a step by case (iv), all steps by case (iii) are also steps by case (iv). Finally, since $\succ_m$ is well-founded by Lemma 2, there can only be finitely many steps by case (ii), which implies that $\succ_j$ is well-founded.

Theorem 5 (Refutation completeness of $\mathcal{I}$ for general clauses). *Let S be a set of clauses and T a set of terms such that the pair $\langle S, T\rangle$ is saturated with respect to $\mathcal{I}$. Then $\square \in S$ if, and only if, Forget(S) is unsatisfiable.*

Proof. Let R denote $\mathit{Forget}(R_S)$. We prove that R^* is a model of $\mathit{Forget}(S)$ by induction on $\succ_j^c$. We derive a contradiction from the existence of a minimal w.r.t. $\succ_j^c$ ground instance $C\theta$ in $\mathit{gnd}_T(S)$ of a clause C in S such that $R^* \not\models \mathit{Forget}(C\theta)$.

1. We first consider the case where C is a positive clause $\rightarrow s\cdot\gamma \simeq t\cdot\gamma, \Delta$, where $\mathit{Forget}((s\cdot\gamma \simeq t\cdot\gamma)\theta)$ is strictly maximal with respect to $\succ^{mul}$ in $\mathit{Forget}(C\theta)$ and w.l.o.g. $\mathit{Forget}((s\cdot\gamma)\theta) \succ \mathit{Forget}((t\cdot\gamma)\theta)$. Since $R^* \not\models \mathit{Forget}(C\theta)$, we know that $C\theta$ has not generated any rule due to one of the following reasons:
   (a) $s \not\succ_r t$ and $t\cdot\gamma$ is not marked on top. In this case we can apply the *mark detection* rule
   $$T \Longrightarrow T \cup \{\mathit{Forget}(t\cdot\gamma)\}.$$
   Now, since $\langle S, T\rangle$ is saturated w.r.t. $\mathcal{I}$, this inference is redundant in T. Thus, $\mathit{Forget}(t\cdot\gamma)$ is an instance of a term t' in T. But then there is an inference π by *marking right*,
   $$\frac{\rightarrow s\cdot\gamma \simeq t\cdot\gamma, \Delta}{\rightarrow s\cdot\gamma \simeq (t\cdot\gamma)^x, \Delta}.$$
   Note that, if t is a variable, then t' is also a variable. Since $t\cdot\gamma$ is not marked on top, we clearly have that $t\cdot\gamma \succ_j (t\cdot\gamma)^x$ by case (i), and hence $C\theta \succ_j D\theta$, where D denotes the conclusion of the previous inference. Now, since $\langle S, T\rangle$ is saturated w.r.t. $\mathcal{I}$, this inference is redundant in S. Therefore, since $C\theta \in \mathit{gnd}_T(S)$, we have that (I) $\mathit{Forget}(\mathit{gnd}_T(S)^{\prec_{red}^c C\theta}) \cup \mathit{Forget}(\mathit{gnd}_T(S)^{\prec_{red}^c D\theta}) \models \mathit{Forget}(D\theta)$ or (II) there is a clause $D'\theta$ in $\mathit{gnd}_T(S)$ of a clause D' in S such that $D'\theta$ is a valid conclusion for the same premises as $\pi\sigma$ and the same set of terms T. If we are in case (I), since $C\theta \succ_j^c D\theta$, $\succ_{red} \subseteq \succ_j$ and, by minimality of $C\theta$, $R^* \models \mathit{Forget}(\mathit{gnd}_T(S)^{\prec_j^c C\theta})$, then we also have that $R^* \models \mathit{Forget}(\mathit{gnd}_T(S)^{\prec_{red}^c C\theta}) \cup \mathit{Forget}(\mathit{gnd}_T(S)^{\prec_{red}^c D\theta})$, and hence $R^* \models \mathit{Forget}(D\theta)$, contradicting $R^* \not\models \mathit{Forget}(C\theta)$. If we are in case (II), then $D'\theta \equiv D\theta$ and hence $D'\theta$ is smaller than $C\theta$ w.r.t. $\succ_j^c$ as well, which implies that $R^* \models \mathit{Forget}(D'\theta)$, contradicting $R^* \not\models \mathit{Forget}(C\theta)$ as before.

(b) $(s \cdot \gamma)\theta$ is reducible by $R^C_S \cup R^C_T$. Then there exists an instance $C'\theta$ of a clause C' of the form $\rightarrow l \cdot \gamma \simeq r \cdot \gamma, \Delta_0$ in $S \cup T_S$, that has generated a rule $l' \cdot \gamma' \rightarrow r' \cdot \gamma'$ reducing $(s \cdot \gamma)\theta$ at some position p. There are the following possibilities:

i. Lifting: $\mathit{Forget}(s \cdot \gamma)|_{p'}$ is a variable x for some prefix p' of p.
Then $p = p' \cdot p''$ for some position p'' such that $x\theta|_{p''} \equiv l' \cdot \gamma'$. Now let θ' be a ground substitution with the same domain as θ but where $x\theta' \equiv x\theta[r' \cdot \gamma']_{p''}$ and $y\theta' \equiv y\theta$ for all other variables y.

A. $l' \cdot \gamma' \rightarrow r' \cdot \gamma' \in R_S$. Then $l' \cdot \gamma' \equiv (l \cdot \gamma)\theta$, $r' \cdot \gamma' \equiv (r \cdot \gamma)\theta$, and either $l \succ_r r$ or $r \cdot \gamma$ is marked on top, which gives us that $C\theta \succ^c_j C\theta'$ by cases (i) and (iii). Moreover, since $R = \mathit{Forget}(R_S)$, we have that $R^* \models \mathit{Forget}(l' \cdot \gamma' \simeq r' \cdot \gamma')$. Then, since $R^* \not\models \mathit{Forget}(C\theta)$, we necessarily have that $R^* \not\models \mathit{Forget}(C\theta')$. We also have that $C\theta' \in \mathit{gnd}_T(S)$, since $C\theta \in \mathit{gnd}_T(S)$ and, moreover, for every marked subterm of the form $(u \cdot \rho)^z$ in $r' \cdot \gamma'$ we have that $\mathit{Forget}(u \cdot \rho)$ is an instance of a term in T by Lemma 21. Therefore $C\theta'$ contradicts the minimality of $C\theta$.

B. $l' \cdot \gamma' \rightarrow r' \cdot \gamma' \in R_T$. In this case, by Lemma 20, $l' \cdot \gamma'$ is not marked on top, $r' \cdot \gamma'$ is marked on top, and $\mathit{Forget}(l' \cdot \gamma') \equiv \mathit{Forget}(r' \cdot \gamma')$. On the one hand, this gives us that $C\theta \succ^c_j C\theta'$ by cases (i) and (ii). On the other hand, we obviously have $R^* \models \mathit{Forget}(l' \cdot \gamma' \simeq r' \cdot \gamma')$. Then a contradiction is derived as in the previous case.

ii. An inference: p is a non-variable position of $\mathit{Forget}(s \cdot \gamma)$.

A. $l' \cdot \gamma' \rightarrow r' \cdot \gamma' \in R_S$. Then $l' \cdot \gamma' \equiv (l \cdot \gamma)\theta$, $r' \cdot \gamma' \equiv (r \cdot \gamma)\theta$, and $\mathit{Forget}((l \cdot \gamma)\theta) \succ \mathit{Forget}((r \cdot \gamma)\theta)$. Therefore there exists an inference π by *paramodulation right*,

$$\frac{\rightarrow l \cdot \gamma \simeq r \cdot \gamma, \Delta_0 \qquad \rightarrow s \cdot \gamma \simeq t \cdot \gamma, \Delta}{(\rightarrow s' \cdot \gamma' \simeq t \cdot \gamma, \Delta'_0, \Delta)\sigma},$$

where $\sigma = \mathit{mgu}(l \cdot \gamma, s \cdot \gamma|_p)$, $\mathit{Forget}(s' \cdot \gamma') \equiv \mathit{Forget}(s \cdot \gamma[r \cdot \gamma]_p)$, $\mathit{Forget}(\Delta'_0)$ is $\mathit{Forget}(\Delta)$, $(s \cdot \gamma)\sigma \succ_{red} (s' \cdot \gamma')\sigma$, or $s \doteq s'$ and $s|_q$ is a variable for some prefix of q of p, and for every equation e in Δ'_0, $\{s \cdot \gamma, t \cdot \gamma\}\sigma \succ^{mul}_{red} e\sigma$ or both sides of e are marked on top. Now we show that the conclusion D of the previous inference has an instance $D\theta$ fulfilling $C\theta \succ^c_j D\theta$. Observe that, if $(s \cdot \gamma)\sigma \succ_{red} (s' \cdot \gamma')\sigma$, then $(s \cdot \gamma)\sigma \succ_j (s' \cdot \gamma')\sigma$ and also $(s \cdot \gamma)\theta \succ_j (s' \cdot \gamma')\theta$, since $\succ_{red} \subseteq \succ_j$ and $\succ_j$ is stable under substitutions. And, if $s \doteq s'$ and $s|_q$ is a variable for some prefix of q of p, then q is a marked position of $s \cdot \gamma$ and, hence, $(s \cdot \gamma)\theta \succ_j (s' \cdot \gamma')\theta$ by case (iii), since $(l \cdot \gamma)\theta \rightarrow (r \cdot \gamma)\theta \in R_S$. W.r.t. the equations e in Δ'_0, either $\{s \cdot \gamma, t \cdot \gamma\}\sigma \succ^{mul}_{red} e\sigma$, and hence $\{s \cdot \gamma, t \cdot \gamma\}\theta \succ^{mul}_j e\theta$, or both sides of e are marked on top. In this latter case, if either $s \cdot \gamma$ or $t \cdot \gamma$ are not marked on top, then we also have that $\{s \cdot \gamma, t \cdot \gamma\}\sigma \succ^{mul}_{red} e\sigma$. If both $s \cdot \gamma$ and $t \cdot \gamma$ are marked on top, then we have that $\{s \cdot \gamma, t \cdot \gamma\}\theta \succ^{mul}_j e\theta$ by case (iv), for the

following reasons: $Forget(\Delta_0\theta)$ is irreducible by $Forget(R_S)$ by Lemma 25; and $Forget((s\cdot\gamma \simeq t\cdot\gamma)\theta) \succ^{mul} Forget(e\theta)$, since $C\theta \succ^c C'\theta$ and $Forget((s\cdot\gamma \simeq t\cdot\gamma)\theta)$ is strictly maximal with respect to $\succ^{mul}$ in $Forget(C\theta)$, which gives us $Forget((s\cdot\gamma \simeq t\cdot\gamma)\theta) \succeq^{mul} Forget((l\cdot\gamma \simeq r\cdot\gamma)\theta) \succ^{mul} Forget(e\theta)$.
Finally, as in case 1a, we derive a contradiction from saturatedness of $\langle S,T\rangle$: We have that this inference is redundant in S. Thus, since $C\theta \in gnd_T(S)$, we have (I) $Forget(gnd_T(S)^{\prec^c_{red} C\theta}) \cup Forget(gnd_T(S)^{\prec^c_{red} D\theta}) \models Forget(D\theta)$ or (II) there is a clause $D'\theta$ in $gnd_T(S)$ of a clause D' in S such that $D'\theta$ is a valid conclusion for the same premises as $\pi\sigma$ and the same set of terms T. If we are in case (I), since $C\theta \succ^c_j D\theta$, $\succ_{red} \subseteq \succ_j$ and, by minimality of $C\theta$, $R^* \models Forget(gnd_T(S)^{\prec^c_j C\theta})$, then we also have that $R^* \models Forget(gnd_T(S)^{\prec^c_{red} C\theta}) \cup Forget(gnd_T(S)^{\prec^c_{red} D\theta})$, and hence $R^* \models Forget(D\theta)$; but, on the other hand, since $R^* \models (l\cdot\gamma)\theta \simeq (r\cdot\gamma)\theta$ and, by Lemma 25, $R^* \not\models Forget(\Delta'_0\theta)$, from $R^* \not\models Forget(C\theta)$ we get that $R^* \not\models Forget(D\theta)$, which is a contradiction. If we are in case (II), then we can assume w.l.o.g. that $D' \equiv D$, since no particular assumption has been made on the marks of D, and hence $D'\theta$ is smaller than $C\theta$ w.r.t. $\succ^c_j$ as well, which implies that $R^* \models Forget(D'\theta)$, contradicting $R^* \not\models Forget(C\theta)$ as before.

B. $l'\cdot\gamma' \to r'\cdot\gamma' \in R_T$. In this case, by Lemma 20, $l'\cdot\gamma'$ is not marked on top, $r'\cdot\gamma'$ is marked on top, and $Forget(l'\cdot\gamma') \equiv Forget(r'\cdot\gamma') \equiv l$ for some term l in T. Since $l'\cdot\gamma'$ is not marked on top and $(s\cdot\gamma)\theta|_p \equiv l'\cdot\gamma'$, then p is a non-marked position of $(s\cdot\gamma)\theta$, and hence of $s\cdot\gamma$. Therefore there is an inference by *marking right*,

$$\frac{\to s\cdot\gamma \simeq t\cdot\gamma, \Delta}{(\to s\cdot\gamma[(s\cdot\gamma|_p)^x]_p \simeq t\cdot\gamma, \Delta)\sigma},$$

where $\sigma = mgu(Forget(s\cdot\gamma|_p), l)$, and whose conclusion D, under the substitution θ, coincides with $(\to s\cdot\gamma[(s\cdot\gamma|_p)^x]_p \simeq t\cdot\gamma, \Delta)\theta$. Moreover, since p is a non-marked position of $s\cdot\gamma$, we have that either $s \gtrdot s'$, or $s \doteq s'$ and $Ran(\gamma) \succ^{mul}_m Ran(\gamma')$, where $s'\cdot\gamma'$ denotes $s\cdot\gamma[(s\cdot\gamma|_p)^x]_p$. Therefore, $s\cdot\gamma \succ_j s\cdot\gamma[(s\cdot\gamma|_p)^x]_p$ either by case (i) or (ii), and hence $C\theta \succ^{mul}_j D\theta$. Then a contradiction is derived as in the previous cases.

(c) $(t\cdot\gamma)\theta$ or $\Delta\theta$ are reducible by $R^C_S \cup R^C_T$ at some non-topmost position. The proof is like that of case 1b.

(d) None of the previous cases applies, and Δ is of the form $u\cdot\gamma \simeq v\cdot\gamma, \Delta'$, where $Forget((s\cdot\gamma)\theta) \equiv Forget((u\cdot\gamma)\theta)$ and $R^* \models Forget((t\cdot\gamma \simeq v\cdot\gamma)\theta)$, that is, the last condition of Definition 13 fails. We consider the following two situations:

i. Either $Forget((t\cdot\gamma)\theta)$ or $Forget((v\cdot\gamma)\theta)$ is reducible by R at a non-topmost position. We assume that $Forget((t\cdot\gamma)\theta)$ is reducible (the proof for the other case is analogous). Then $Forget((t \cdot \gamma)\theta)|_p \equiv Forget(l \cdot \delta)$ for some rule $l \cdot \delta \to r \cdot \delta$ in R_S and some position $p \neq \lambda$. In fact, since $\rhd \subseteq \succ$, no rule generated by a greater clause can reduce $Forget((t \cdot \gamma)\theta)$ at a non-topmost position, and hence $l \cdot \delta \to r \cdot \delta \in R_S^C$. Now recall that $(t \cdot \gamma)\theta$ is irreducible by R_S^C at non-topmost positions, since none of the previous cases applies. Therefore, the only possibility is that there are some positions q and p' such that $q = p \cdot p'$ and $Forget((t \cdot \gamma)\theta)|_q \equiv Forget((l \cdot \delta)\theta)|_{p'}$, and either q is a marked position of $(t \cdot \gamma)\theta$ and p' is a non-marked position of $(l \cdot \delta)\theta$, or q is a non-marked position of $(t \cdot \gamma)\theta$ and p' is a marked position of $(l \cdot \delta)\theta$.

- If q is a marked position of $(t \cdot \gamma)\theta$ then, since $C\theta \in gnd_T(S)$ and, moreover, for every marked term of the form $(l' \cdot \delta')^x$ in occurring in S we have that $Forget(l' \cdot \delta')$ is an instance of a term in T, we must have that $Forget((t \cdot \gamma)\theta|_q)$ is an instance of a term t in T. But then also $Forget((l \cdot \delta)\theta|_{p'})$ is an instance of t and, therefore, by Lemma 22 we have that $(l \cdot \delta)\theta|_{p'}$ is marked on top, contradicting that p' is a non-marked position of $(l \cdot \delta)\theta$.
- If q is a non-marked position of $(t \cdot \gamma)\theta$ and p' is a marked position of $(l \cdot \delta)\theta$, then we have that $(l \cdot \delta)\theta|_{p'} \equiv t$ for some term t in T by Lemma 21 and, as in case 1b, either the lifting argument applies or there is an inference by *marking right*, and hence a contradiction is derived as well.

ii. $Forget((t \cdot \gamma)\theta)$ and $Forget((v \cdot \gamma)\theta)$ are irreducible by R at non-topmost positions. In this case we consider the following inference by *equality factoring*,

$$\frac{\to s \cdot \gamma \simeq t \cdot \gamma, u \cdot \gamma \simeq v \cdot \gamma, \Delta'}{(t' \cdot \gamma' \simeq v' \cdot \gamma' \to s \cdot \gamma \simeq t \cdot \gamma, \Delta')\sigma},$$

where $\sigma = mgu(s \cdot \gamma, u \cdot \gamma)$, $Forget(t' \cdot \gamma') \equiv Forget(t \cdot \gamma)$, $Forget(v' \cdot \gamma') \equiv Forget(v \cdot \gamma)$, and $(u \cdot \gamma)\sigma \succ_{red} (t' \cdot \gamma')\sigma$ or $t' \cdot \gamma'$ is marked on top, and $(u \cdot \gamma)\sigma \succ_{red} (v' \cdot \gamma')\sigma$ or $v' \cdot \gamma'$ is marked on top. We clearly have that the conclusion D of the previous inference coincides with $(t' \cdot \gamma' \simeq v' \cdot \gamma' \to s \cdot \gamma \simeq t \cdot \gamma, \Delta')\theta$ under the substitution θ.
Now we show that $(u \cdot \gamma)\theta \succ_j (t' \cdot \gamma')\theta$ (the proof for $(u \cdot \gamma)\sigma \succ_j (v' \cdot \gamma')\theta$ is analogous). If $(u \cdot \gamma)\sigma \succ_{red} (t' \cdot \gamma')\sigma$ then $(u \cdot \gamma)\theta \succ_j (t' \cdot \gamma')\theta$, since $\succ_{red} \subseteq \succ_j$ and $\succ_j$ is stable under substitutions. If $t' \cdot \gamma$ is marked on top and $u \cdot \gamma$ is not, then $u \cdot \gamma \succ_{red} t' \cdot \gamma'$ and we conclude as before. If both $u \cdot \gamma$ and $t' \cdot \gamma'$ are marked on top, then we have that $(u \cdot \gamma)\theta \succ_j (t' \cdot \gamma')\theta$ by case (iv), since $Forget((u \cdot \gamma)\theta) \succ Forget((t' \cdot \gamma')\theta)$, and $Forget((t' \cdot \gamma')\theta)$ is assumed to be irreducible by R at non-topmost positions.
Then, since $\{(u \cdot \gamma)\theta, (v \cdot \gamma)\theta\} \succ_j^{mul} \{(t' \cdot \gamma')\theta, (t' \cdot \gamma')\theta, (v' \cdot \gamma')\theta, (v' \cdot \gamma')\theta\}$, we have that $C\theta \succ_j^c D\theta$, and from this fact a contradiction can be derived as in previous cases.

2. If C is a positive clause $\rightarrow s \cdot \gamma \simeq t \cdot \gamma, \Delta$, where *Forget*$((s \cdot \gamma \simeq t \cdot \gamma)\theta)$ is maximal but not strictly maximal with respect to $\succ^{mul}$ in *Forget*$(C\theta)$, then condition 1c of Definition 13 fails. If $s \not\succ_r t$ and $t \cdot \gamma$ is not marked on top, the reasoning of case 1a applies. If $(s \cdot \gamma \simeq t \cdot \gamma)\theta$ is reducible by R_S at a non-topmost position, then the same reasoning as in case 1b applies. Otherwise, the reasoning of case 1d applies.
3. If C is clause $\Gamma, \underline{s \cdot \gamma \simeq t \cdot \gamma} \rightarrow \Delta$ where *Forget*$((s \cdot \gamma)\theta) \equiv$ *Forget*$((t \cdot \gamma)\theta)$, then there is an inference by *equality resolution*,

$$\frac{\Gamma, \underline{s \cdot \gamma \simeq t \cdot \gamma} \rightarrow \Delta}{(\Gamma \rightarrow \Delta)\sigma},$$

where $\sigma = mgu(s \cdot \gamma, t \cdot \gamma)$. Then we clearly have that $C\theta \succ_j^c D\theta$, where D denotes the conclusion of the previous inference. This fact, together with the saturatedness assumption, leads to a contradiction as in the other cases.
4. If C is a clause $\Gamma, \underline{s \cdot \gamma \simeq t \cdot \gamma} \rightarrow \Delta$ where *Forget*$((s \cdot \gamma)\theta) \not\equiv$ *Forget*$((t \cdot \gamma)\theta)$, then assume w.l.o.g. that *Forget*$((s \cdot \gamma)\theta) \succ$ *Forget*$((t \cdot \gamma)\theta)$. Let s' denote *Forget*$((s \cdot \gamma)\theta)$ and t' denote *Forget*$((t \cdot \gamma)\theta)$. Then, since $R^* \not\models$ *Forget*$(C\theta)$, we have that $R^* \models s' \simeq t'$ and, since R is convergent by Lemma 24, there must be a rewrite proof of $s' \simeq t'$ with R, that is, s' and t' must rewrite into the same normal form with R. This implies that either t' is reducible at a non-topmost position or else s' is reducible. (Note that it cannot be the case that the only possible reduction step on $s' \simeq t'$ is at the topmost position of t'. By such a step, a new term t'' is obtained with $t' \succ t''$ and t'' again irreducible at non-topmost positions by Lemma 23; since $s' \succ t'$, such a sequence of topmost steps on t' can never produce s'). We consider the case where s' is reducible. (The other one is analogous.) Then there is an instance of a clause $\rightarrow l \cdot \gamma \simeq r \cdot \gamma, \Delta_0$ in S that has generated a rule $(l \cdot \gamma)\theta \rightarrow (r \cdot \gamma)\theta$ in R_S, such that *Forget*$((s \cdot \gamma)\theta)|_p \equiv$ *Forget*$((l \cdot \gamma)\theta)$ for some position p. We have the following possibilities:
   (a) $(s \cdot \gamma)\theta|_p \equiv (l \cdot \gamma)\theta$. Then either the lifting argument applies like in case 1(b)iA, or else there is an inference by *paramodulation left*, which leads to a contradiction analogously to case 1(b)iiA.
   (b) $(s \cdot \gamma)\theta|_p \not\equiv (l \cdot \delta)\theta$. Then, analogously to case 2b of Theorem 2, either the lifting argument applies or else there is an inference by *marking left*, and a contradiction is derived respectively as in cases 1(b)iB and 1(b)iiB of this theorem.

Theorem 6. *If $\langle S_0, T_0\rangle, \langle S_1, T_1\rangle, \ldots$ is a fair derivation with respect to $\mathcal{I}$, then Forget(S_0) is unsatisfiable if, and only if, $\Box \in S_j$ for some j. Furthermore, if $\langle S_0, T_0\rangle, \langle S_1, T_1\rangle, \ldots \langle S_n, T_n\rangle$ is fair, then S_n is saturated and Forget(S_n) is logically equivalent to Forget(S_0).*

Proof. By Lemma 19 we have that $\langle S_\infty, T_\infty\rangle$ is saturated with respect to $\mathcal{I}$. Then, by Theorem 5, we have that $\Box \in S_\infty$ if *Forget*(S_∞) is unsatisfiable. Since, by definition of derivation, *Forget*(S_0) is logically equivalent to all *Forget*(S_i)

with $i \geq 0$ and, by Lemma 18, to $Forget(S_\infty)$ as well, $Forget(S_0)$ is unsatisfiable if, and only if, $\Box \in S_\infty$ and hence $\Box \in S_j$ for some j. Finally if $\langle S_0, T_0\rangle, \langle S_1, T_1\rangle, \ldots \langle S_n, T_n\rangle$ is fair then $S_\infty = S_n$ and hence S_n is saturated.

10 Conclusions

In this paper we have improved on the results of [BGNR03] on completeness of ordered paramodulation with non-monotonic orderings, by adapting those refutation complete inference systems so that they are compatible with powerful redundancy notions like demodulation.

We have proposed some inference systems (for equations, Horn clauses and general first order clauses) that work with a pair of orderings: a west ordering $\succ$, which is used in the inferences, and a reduction ordering $\succ_r$, included in $\succ$, which is used in the redundancy notions. Our method is based on adding some marks that block terms for redundancy (but not for inferences). In it, if there are many equations that cannot be handled by the reduction ordering, then many terms will become blocked and we will have almost no redundancy, but the same amount of inferences by paramodulation as in [BGNR03]. On the other hand, if many equations can be handled by the reduction ordering, then few terms will be blocked and we will have as much redundancy as in the paramodulation calculi for total reduction orderings. Therefore now it becomes interesting to obtain experimental results to see how do these inference systems behave in practice.

Using these results in the same way as in [BGNR03], we have obtained a Knuth-Bendix completion procedure, compatible with simplification by rewriting, that finds a convergent TRS for a given set of equations E and a (possibly non-totalizable) reduction ordering $\succ_r$ whenever it exists.

As a final remark, we want to note that a way to obtain, in practice, the orderings $\succ$ and $\succ_r$, is to define the reduction ordering $\succ_r$ with the *monotonic semantic path orderings* [BFR00], since then it is directly included, by definition, in a *semantic path ordering* [KL80], which can be used as the west ordering $\succ$.

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