On the Completeness of Arbitrary Selection Strategies for Paramodulation*

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Abstract. A crucial way for reducing the search space in automated deduction are the so-called *selection strategies*: in each clause, the subset of *selected* literals are the only ones involved in inferences.
For first-order Horn clauses without equality, resolution is complete with an arbitrary selection of one single literal in each clause [dN96].
For Horn clauses with built-in equality, i.e., paramodulation-based inference systems, the situation is far more complex. Here we show that if a paramodulation-based inference system is complete with eager selection of negative equations and, moreover, it is compatible with equality constraint inheritance, then it is complete with arbitrary selection strategies.
A first important application of this result is the one for paramodulation wrt. non-monotonic orderings, which was left open in [BGNR99].

1 Introduction

Deduction with equality is fundamental in mathematics, logics and many applications of formal methods in computer science. During the last two decades this field has importantly progressed through new Knuth-Bendix-like completion techniques and their extensions to ordered paramodulation for first-order clauses. These techniques have lead to important results on theorem proving in first-order logic with equality [HR91,BDH86,BD94,BG94] (that have been applied to state-of-the-art theorem provers like Spass [Wei97]), results on logic-based complexity and decidability analysis [BG96,Nie98], on deduction with *constrained clauses* [KKR90,NR95], and on many other applications like inductive theorem proving, symbolic constraint solving, or equational-logic programming (see [NR01] for a recent survey).

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A crucial way for reducing the search space in automated deduction are the so-called *selection strategies*. In such strategies the possible inferences between clauses are restricted by allowing only inferences involving *selected* literals. This selection can be done in several different ways. Well-known examples of selection strategies are the *maximal* strategies for a given atom ordering. For example, in a maximal resolution strategy, a (ground) inference between $A \vee C$ and $\neg A \vee D$ is performed only if A is larger in the given atom ordering than all other atoms in C and D. Another well-known selection strategy is the so-called *eager* negative selection strategy, where in each clause a single negative literal is selected whenever there is any. This leads to the so-called *positive* strategies (positive unit strategies in the Horn case) because always the left premise of each (resolution or paramodulation) inference is a positive (unit) clause. These strategies are usually easy to prove complete, but sometimes they are not very efficient, because, roughly speaking, one enumerates all solutions of its conditions before using the positive information of a clause [Der91].

Another well-known way for reducing the search space is constraint inheritance. The semantics of a clause C with a constraint T, written $C \mid T$, is simply the set of all ground instances $C\sigma$ of C such that σ is a solution of T. For example, if $=$ denotes syntactic equality of terms, the constrained clause $P(x) \mid x = f(y)$ denotes all ground atoms of the form $P(f(t))$. Constraints are closely related to *basic* strategies [BGLS95,NR95], where no inferences have to be computed on subterms generated in unifiers of ancestor inference steps. In fact, an inference system with a basic strategy can be modelled as working with constrained clauses, inheriting them and simply storing the unification restrictions in the constraint part (i.e., without applying the constraints to the clause part).

For first-order Horn clauses without equality, resolution is complete with an arbitrary selection of one single literal in each clause ([dN96], Theorem 6.7.4). For Horn clauses with built-in equality, i.e., paramodulation-based inference systems, the situation is far more complex. In [Lyn97] some positive and negative results are given for the case where a total *reduction* (well-founded, monotonic) ordering on ground terms is given. Then arbitrary selection strategies are compatible with superposition[1] (that is, paramodulation involving only maximal sides of equations). Also conditions for eliminating *redundant* clauses are given in [Lyn97], and counter examples indicating the limitations for doing so. For example, in certain circumstances the elimination of tautologies can lead to incompleteness.

Here we show a more general result for Horn clauses with equality, namely that, if a paramodulation-based inference system is complete with eager selection of negative equations and, moreover, it is compatible with equality constraint inheritance (like, in particular, it happens for superposition), then it is complete with arbitrary selection strategies.

Our completeness result is based on transformations of proof trees. Its generality allows us to obtain directly the completeness of arbitrary selection strategies

[1] About two years ago we contacted the author of [Lyn97] about some severe flaws in his completeness proofs, which he agreed to fix, although to our knowledge this has not happened yet.

for other inference systems, apart from the one of superposition with total reduction orderings. A first important application of our result is the one for paramodulation with non-monotonic orderings of [BGNR99], where the completeness of strategies different from eager negative selection was left open. There, techniques for dropping the monotonicity requirement were introduced, with the only properties required for the ordering being well-foundedness and the subterm property. However, the inference system of [BGNR99] still required the eager selection of negative equations. In Section 4 we show that those results are compatible with equality constraint inheritance and hence with the *basic* strategy, thus further restricting the search space. Therefore, our transformation method is applicable, and we obtain the completeness of the same inference system but with arbitrary selection strategies.

The structure of the paper is the following. In section 2 we give some basic notions on terms, equations and clauses. In section 3 we present our transformation method for proofs, and our main result on completeness of arbitrary selection strategies. In section 4 we apply this new technique to the case of the inference system of [BGNR99]. Finally, in section 5 we discuss some possible extensions of our technique, giving counter examples indicating the limitations for some of the extensions.

2 Basic Notions

We use the standard definitions of [DJ90]: $T(\mathcal{F},\mathcal{X})$ ($T(\mathcal{F})$) is the set of (ground) terms over $\mathcal{F}$, the subterm of t at *position* p is denoted $t|_p$, the result of replacing $t|_p$ by s in t is denoted $t[s]_p$, and syntactic equality of terms is denoted by $\equiv$.

A *substitution* σ is a mapping from variables to terms. It can be extended to a function from terms to terms (atoms, clauses) in the usual way: using a postfix notation, $t\sigma$ denotes the result of simultaneously replacing in t every $x \in Dom(\sigma)$ by $x\sigma$. A substitution σ is *ground* if its range is $\mathcal{T}(\mathcal{F})$. Here we suppose that ground substitutions σ are also *grounding*: $t\sigma$ is ground for all t.

An *equation* is a multiset $\{s,t\}$, denoted $s \simeq t$ or, equivalently, $t \simeq s$.

A first-order clause is a pair of finite multisets of equations Γ (the *antecedent*) and Δ (the *succedent*), denoted by $\Gamma \rightarrow \Delta$. The empty clause $\Box$ is a clause where both Γ and Δ are empty. For dealing with non-equality predicates, atoms A can be expressed by equations $A \simeq true$ where $true$ is a new symbol.

The semantics of a clause C with a constraint T, written $C \mid T$, is simply the set of all ground instances $C\sigma$ of C such that σ is a solution of T.

If $\rightarrow$ is binary relation, then $\leftarrow$ is its inverse, $\leftrightarrow$ is its symmetric closure, $\rightarrow^+$ is its transitive closure and $\rightarrow^*$ is its reflexive-transitive closure. We write $s \rightarrow^! t$ if $s \rightarrow^* t$ and there is no t' such that $t \rightarrow t'$. Then t is called *irreducible* and a *normal form* of s (wrt. $\rightarrow$). The relation $\rightarrow$ is *well-founded* or *terminating* if there exists no infinite sequence $s_1 \rightarrow s_2 \rightarrow \ldots$ and it is *confluent* or *Church-Rosser* if the relation $\leftarrow^* \circ \rightarrow^*$ is contained in $\rightarrow^* \circ \leftarrow^*$. It is *locally confluent* if $\leftarrow \circ \rightarrow \;\subseteq\; \rightarrow^* \circ \leftarrow^*$. By Newman's lemma, terminating locally-confluent relations are confluent. A relation $\rightarrow$ on terms is *monotonic* if $s \rightarrow t$ implies $u[s]_p \rightarrow u[t]_p$ for all terms s, t and u and positions p. A *congruence* is a reflexive, symmetric, transitive and monotonic relation on terms.

A *rewrite rule* is an ordered pair of terms (s, t), written $s \rightarrow t$, and a set of rewrite rules R is a *term rewrite system* (TRS). The rewrite relation with R on $T(\mathcal{F}, \mathcal{X})$, denoted $\rightarrow_R$, is the smallest monotonic relation such that $l\sigma \rightarrow_R r\sigma$ for all $l \rightarrow r \in R$ and all σ, and if $s \rightarrow_R t$ then we say that *s rewrites into t* with R. R is called terminating, confluent, etc. if $\rightarrow_R$ is. A rewrite system R is *convergent* if it is confluent and terminating; then every term t has a unique normal form wrt. $\rightarrow_R$, denoted by $nf_R(t)$, and $s \simeq t$ is a logical consequence of R (where R is seen as a set of equations) iff $nf_R(s) = nf_R(t)$.

Let R be a set of ground equations or rewrite rules. Then the congruence $\leftrightarrow_R^*$ defines an equality Herbrand interpretation denoted by R^*, where the only predicate $\simeq$ is interpreted by $s \simeq t$ iff $s \leftrightarrow_R^* t$. We write $s = t \in R^*$ if $s \leftrightarrow_R^* t$. R^* satisfies (is a model of) a ground clause $\Gamma \rightarrow \Delta$, denoted $R^* \models \Gamma \rightarrow \Delta$, if $R^* \not\supseteq \Gamma$ or $R^* \cap \Delta \neq \emptyset$. The empty clause $\square$ is hence satisfied by no interpretation. R^* satisfies a set of clauses S, denoted by $R^* \models S$, if it satisfies every clause in S.

A (strict partial) *ordering* on $T(\mathcal{F}, \mathcal{X})$ is an irreflexive transitive relation $\succ$. It is a *reduction* ordering if it is well-founded and monotonic, and stable under substitutions: $s \succ t$ implies $s\sigma \succ t\sigma$ for all substitutions σ. It fulfils the *subterm property* if $\succ \supseteq \rhd$, where $\rhd$ denotes the strict subterm ordering.

3 The transformation method

In the following we deal with inference systems that are based on some selection strategy. A selection strategy is a function from ground clauses to non-empty sets of literals, such that the selected literals for a clause appear in the clause. An inference between two ground clauses is allowed only if the literals involved in the inference are selected. As usual, a non-ground inference represents all its ground instances fulfiling the required conditions. In our case, a non-ground inference is allowed only if, for some ground instance of the inference, the involved literals are selected.

Our hypothesis in this section is that we have at hand a paramodulation-based inference system $\mathcal{N}$ for first-order Horn clauses, which is compatible with equality constraint inheritance and complete with a concrete strategy with eager selection of negative equations. Let $\mathcal{N}$ consist of the following inference rules:

paramodulation right:

$$\frac{\rightarrow l \simeq r \mid T_1 \qquad \rightarrow s \simeq t \mid T_2}{\rightarrow s[r]_p \simeq t \mid s|_p = l \wedge T_1 \wedge T_2} \qquad \text{if } s|_p \notin \mathcal{X}$$

paramodulation left:

$$\frac{\rightarrow l \simeq r \mid T_1 \qquad \Gamma, \underline{s \simeq t} \rightarrow \Delta \ \mid T_2}{\Gamma, s[r]_p \simeq t \rightarrow \Delta \mid s|_p = l \wedge T_1 \wedge T_2} \qquad \text{if } s|_p \notin \mathcal{X}$$

equality resolution:

$$\frac{\Gamma, \underline{s \simeq t} \to \Delta \mid T}{\Gamma \to \Delta \mid s = t \wedge T}$$

where the equations written underlined must belong to the set of selected literals. Here $=$ is interpreted as the syntactic equality relation $\equiv$ when dealing with instances. That is, we forbid those instances of the conclusion that correspond to ground inferences between instances of the premises for which the constraints do not hold.

Our aim is to prove completeness of the following inference system $\mathcal{A}$, which is a modification of $\mathcal{N}$ allowing an arbitrary selection strategy where a single arbitrary literal is selected in each ground clause:

paramodulation right:

$$\frac{\Gamma_1 \to \underline{l \simeq r} \mid T_1 \qquad \Gamma_2 \to \underline{s \simeq t} \mid T_2}{\Gamma_1, \Gamma_2 \to s[r]_p \simeq t \mid s|_p = l \wedge T_1 \wedge T_2} \qquad \text{if } s|_p \notin \mathcal{X}$$

paramodulation left:

$$\frac{\Gamma_1 \to \underline{l \simeq r} \mid T_1 \qquad \Gamma_2, \underline{s \simeq t} \to \Delta \mid T_2}{\Gamma_1, \Gamma_2, s[r]_p \simeq t \to \Delta \mid s|_p = l \wedge T_1 \wedge T_2} \qquad \text{if } s|_p \notin \mathcal{X}$$

equality resolution:

$$\frac{\Gamma, \underline{s \simeq t} \to \Delta \mid T}{\Gamma \to \Delta \mid s = t \wedge T}$$

In order to prove the completeness of $\mathcal{A}$ we will proceed as follows. Assume S is a set of constrained clauses that is closed under $\mathcal{A}$. Furthermore, let P be a proof by $\mathcal{N}$ deriving the empty clause from S. Then we will show that if P is non-trivial, i.e., it has more than zero steps, then there exists another proof by $\mathcal{N}$ from S of the empty clause with a smaller number of steps. By induction on this proof transformation process, it follows that the empty clause belongs to S.

Let S be a set of constrained clauses and let $C \mid T$ be a constrained clause that is in the closure of S wrt. $\mathcal{N}$. Then, as usual, the proof by $\mathcal{N}$ of $C \mid T$ from S can be expressed as a tree rooted by $C \mid T$, and whose leaves are in S. Now assume T is satisfiable, and let σ be a ground solution of T. Furthermore, σ can be taken such that its domain contains all variables ocurring in the proof. Therefore we can deal with ground proofs where the constraints are replaced by their solution σ (where a ground substitution σ itself is seen as an equality constraint): by a (ground) $\mathcal{N}$*-proof* P of $C \mid \sigma$ from S we mean a proof tree by $\mathcal{N}$, whose nodes are clauses of the form $D \mid \sigma$, and whose leaves are clauses $D' \mid \sigma$ where $D' \mid T'$ is in S and $\sigma \models T'$. By $\mathit{steps}(P)$ we refer to its number of proof steps (or, equivalently, to its number of non-leaf nodes). The following is an example of an $\mathcal{N}$-proof.

Example 1.

$$\dfrac{\rightarrow c \simeq a \mid x = a \qquad \dfrac{x \simeq a \rightarrow b \simeq c \mid x = a}{\rightarrow b \simeq c \mid x = a}}{\rightarrow b \simeq a \mid x = a}$$

When dealing with $\mathcal{N}$-proofs, we will frequently speak about its rightmost leaf ($x \simeq a \rightarrow b \simeq c \mid x = a$ in the example), its rightmost inner node ($\rightarrow b \simeq c \mid x = a$), its rightmost step (the inference obtaining $\rightarrow b \simeq c \mid x = a$ from $x \simeq a \rightarrow b \simeq c \mid x = a$), and its rightmost path (the nodes $x \simeq a \rightarrow b \simeq c \mid x = a$, $\rightarrow b \simeq c \mid x = a$, $\rightarrow b \simeq a \mid x = a$).

An $\mathcal{N}$-proof is called *antecedent elimination of Γ* if its rightmost leaf is of the form $\Gamma \rightarrow \Delta \mid \sigma$, its root is $\rightarrow \Delta \mid \sigma$, and no node on its rightmost path is obtained by a paramodulation-right step. Note that in an antecedent elimination it is relevant which is Δ: for some Δ', the same proof (up to replacements of Δ by Δ') may not be correct, since changing the clause may result in a change in which negative equations are selected. But, as we will see in lemma 3, this change is possible by modifying the $\mathcal{N}$-proof.

3.1 Completeness proof

In the given proofs, the substitution part $\mid \sigma$ of the clauses is omitted in order to improve readability (proofs of fusion lemma, separation lemma, and their general versions are given in appendix A).

Lemma 1. (fusion lemma) *Let P_1 and P_2 be two antecedent elimination $\mathcal{N}$-proofs of Γ_1 and Γ_2 respectively.*

Then for an arbitrary Δ, there exists an antecedent elimination $\mathcal{N}$-proof P such that its rightmost leaf is $\Gamma_1, \Gamma_2 \rightarrow \Delta$. Moreover, $steps(P) = steps(P_1) + steps(P_2)$, and every non-rightmost leaf of P is a non-rightmost leaf of P_1 or of P_2.

In the following, the $\mathcal{N}$-proof P built as in the previous lemma will be called the *fusion* of P_1 and P_2.

Lemma 2. *(separation lemma) Let P be an antecedent elimination $\mathcal{N}$-proof of Γ_1, Γ_2.*

Then, there exist two antecedent elimination $\mathcal{N}$-proofs P_1 and P_2 of Γ_1 and Γ_2 respectively. Moreover $steps(P) = steps(P_1) + steps(P_2)$, and all the non-rightmost leaves of P_1 or of P_2 are non-rightmost leaves of P.

Lemma 3. *Let P be an antecedent elimination $\mathcal{N}$-proof of Γ.*

Then, for all Δ, there exists an antecedent elimination $\mathcal{N}$-proof P' such that its rightmost leaf is $\Gamma \rightarrow \Delta$. Moreover, $steps(P) = steps(P')$.

Proof. By applying the fusion lemma to P and the empty proof we obtain the desired P'. $\square$

Lemma 4. *(general fusion lemma) Let P_1 be an antecedent elimination $\mathcal{N}$-proof of Γ_1. Let P_2 be an $\mathcal{N}$-proof of $\rightarrow \Delta$ with rightmost leaf $\Gamma_2 \rightarrow \Delta_2$ (therefore a subtree in its rightmost path is antecedent elimination, i.e. the antecedent Γ_2 is eliminated in P_2, but after this elimination, some paramodulation right inferences can be made on the rightmost path).*

Then there exists an $\mathcal{N}$-proof P of $\rightarrow \Delta$ such that its rightmost leaf is $\Gamma_1, \Gamma_2 \rightarrow \Delta_2$. Moreover $steps(P) = steps(P_1) + steps(P_2)$, and every non-rightmost leaf of P is a non-rightmost leaf of P_1 or of P_2.

Lemma 5. *(general separation lemma) Let P be an $\mathcal{N}$-proof of $\rightarrow \Delta$ such that its rightmost leaf is $\Gamma_1, \Gamma_2 \rightarrow \Delta'$.*

Then, there exist two $\mathcal{N}$-proofs P_1 and P_2 such that P_1 is an antecedent elimination proof of Γ_1 and P_2 is an $\mathcal{N}$-proof of $\rightarrow \Delta$ with rightmost leaf $\Gamma_2 \rightarrow \Delta'$. Moreover $steps(P) = steps(P_1) + steps(P_2)$, and all the non-rightmost leaves of P_1 or of P_2 are non-rightmost leaves of P.

Lemma 6. *Let S be a set of clauses closed under $\mathcal{A}$, and let P be an $\mathcal{N}$-proof of $\rightarrow \Delta$ from S.*

Then there exists an antecedent elimination $\mathcal{N}$-proof P' of $\rightarrow \Delta$ from S whose rightmost leaf is of the form $\Gamma \rightarrow \Delta$, where $steps(P') \leq steps(P)$ and, if Γ is non-empty, then Δ is non-empty and its equation is the selected one in $\Gamma \rightarrow \Delta$ by $\mathcal{A}$.

Proof. We will proceed by induction on $steps(P)$. Let the rightmost leaf of P be of the form $\Gamma_1 \rightarrow \Delta_1$. There are several cases to be considered:

0. If Γ_1 and Δ_1 are both empty, then P has no steps and P' can be P itself.

1. One of the selected equations of $\Gamma_1 \rightarrow \Delta_1$ by $\mathcal{A}$ is Δ_1. We consider two possibilities depending on whether some paramodulation right inference is made or not on the rightmost path of P.

1a. In the case that no paramodulation right inference is made on the rightmost path of P, we have that Δ and Δ_1 coincide and P is antecedent elimination. Therefore the P' we are looking for is directly P.

1b. Suppose now there are some paramodulation right inferences on the rightmost path of P. Then the highest one is of the form:

$$\frac{\rightarrow \Delta_2 \qquad \rightarrow \Delta_1}{\rightarrow \Delta_3}$$

Let P_1 be the subproof rooted by $\rightarrow \Delta_1$. It is an antecedent elimination $\mathcal{N}$-proof. Let P_2 be the subproof rooted by $\rightarrow \Delta_2$. Let P_3 be like P, but where P_1 and P_2 are removed (the rightmost leaf of P is $\rightarrow \Delta_3$). Then we have $steps(P) = steps(P_1)+steps(P_2)+steps(P_3)+1$. By applying induction hypothesis to P_2, we obtain an antecedent elimination $\mathcal{N}$-proof P_2' of $\rightarrow \Delta_2$ from S s.t. its rightmost leaf is of the form $\Gamma_2 \rightarrow \Delta_2$, where Δ_2 is selected by $\mathcal{A}$, and $steps(P_2') \leq$

$steps(P_2)$. If we apply the fusion lemma to P_2' and P_1, we obtain an antecedent elimination $\mathcal{N}$-proof P_4 of $\rightarrow \Delta_3$, where its rightmost leaf is $\Gamma_1, \Gamma_2 \rightarrow \Delta_3$, and $steps(P_4) = steps(P_2') + steps(P_1) \leq steps(P_2) + steps(P_1)$. Now let P_5 be the $\mathcal{N}$-proof formed by P_3, and where above its rightmost leaf we insert P_4. We have $\rightarrow \Delta$ in the root of P_5, and $steps(P_5) = steps(P_3) + steps(P_4) \leq steps(P_3) + steps(P_2) + steps(P_1) < steps(P)$ and the rightmost leaf of P_5 is $\Gamma_2, \Gamma_1 \rightarrow \Delta_3$. Moreover, every non-rightmost leaf of P_5 is from S: all the non-rightmost leaves of P_3 are from S, and, since P_4 is the fusion of P_2' and P_1, then, its non-rightmost leaves are from S too. But also, the rightmost leaf of P_5 is from S, since the next inference is a $\mathcal{A}$-inference from S:

$$\frac{\Gamma_2 \rightarrow \underline{\Delta_2} \qquad \Gamma_1 \rightarrow \underline{\Delta_1}}{\Gamma_2, \Gamma_1 \rightarrow \Delta_3}$$

Then the $\mathcal{N}$-proof P' we are looking for is the one obtained by applying the induction hypothesis to P_5.

2. Assume now that Γ_1 is of the form Γ_{11}, e and e is a selected equation of $\Gamma_1 \rightarrow \Delta_1$ by $\mathcal{A}$. We apply the general separation lemma to P and we obtain two $\mathcal{N}$-proofs P_1 and P_2 s.t. P_1 is an antecedent elimination proof of e and P_2 is an $\mathcal{N}$-proof of $\rightarrow \Delta$ with rightmost leaf $\Gamma_{11} \rightarrow \Delta_1$. Moreover $steps(P) = steps(P_1) + steps(P_2)$, and all the non-rightmost leaves of P_1 or of P_2 are non-rightmost leaves of P.

We distinguish two cases depending on wether the rightmost step of P_1 is an equality resolution or a paramodulation left inference on e:

2a. If it is an equality resolution step, it is of the form:

$$\frac{e \rightarrow}{\rightarrow}$$

Then P_1 consists in only this step, and we have the $\mathcal{A}$-inference

$$\frac{\underline{e}, \Gamma_{11} \rightarrow \Delta_1}{\Gamma_{11} \rightarrow \Delta_1}$$

and hence $\Gamma_{11} \rightarrow \Delta_1$ is in S. Therefore the $\mathcal{N}$-proof P' we are looking for is the one obtained by applying the induction hypothesis to P_2.

2b. If the rightmost step of P_1 is a paramodulation left inference, it is of the form:

$$\frac{\rightarrow \Delta_3 \qquad e \rightarrow}{e' \rightarrow}$$

Let P_3 be the subproof of P_1 rooted by $\rightarrow \Delta_3$. Let P_1' be like P_1 but where the subproof P_3 and the rightmost leaf are removed (the rightmost leaf of P_1' is $e' \rightarrow$). Note that all the non-rightmost leaves of P_1' are clauses from S. We have $steps(P_1) = steps(P_3) + steps(P_1') + 1$. We apply induction hypothesis to P_3, and we obtain an antecedent elimination $\mathcal{N}$-proof P_3' of $\rightarrow \Delta_3$ from S s.t.

its rightmost leaf is of the form $\Gamma_3 \to \Delta_3$, where Δ_3 is selected by $\mathcal{A}$. Now, we apply the fusion lemma to P_3' and P_1', and we get an $\mathcal{N}$-proof P_4 s.t. is antecedent elimination of Γ_3, e', and $steps(P_4) = steps(P_3') + steps(P_1')$. Applying now the general fusion lemma to P_4 and P_2 we obtain an $\mathcal{N}$-proof P_5 s.t. its rightmost leaf is $\Gamma_{11}, e', \Gamma_3 \to \Delta_1$, $steps(P_5) = steps(P_3') + steps(P_1') + steps(P_2) < steps(P)$ and all the non-rightmost leaves of P_5 are non-rightmost leaves of P_3' or of P_1' or of P_2 and, hence, from S. But also the rightmost leaf of P_5 is a clause from S, since the next inference is an $\mathcal{A}$-inference from S:

$$\frac{\Gamma_3 \to \underline{\Delta_3} \qquad \Gamma_{11}, \underline{e} \to \Delta_1}{\Gamma_3, \Gamma_{11}, e' \to \Delta_1}$$

Therefore the $\mathcal{N}$-proof P' we are looking for is the one obtained by applying the induction hypothesis to P_5. $\square$

Theorem 1. *(Completeness theorem) Let S_0 be an unconstrained set of clauses, and let S be the closure of S_0 under $\mathcal{A}$. If S_0 is unsatisfiable, then $\square \in S$.*

Proof. By completeness of $\mathcal{N}$ there is an $\mathcal{N}$-proof of $\square$ from S. Then, applying lemma 6 to the case where Δ is empty gives us a trivial $\mathcal{N}$-proof of $\square$, since Δ can not be selected by $\mathcal{A}$ and, hence, Γ must be empty. Consequently $\square \in S$. $\square$

4 Application to paramodulation with respect to non-monotonic orderings

Here we prove the refutational completeness of the inference system presented in [BGNR99] with added equality constraint inheritance, for the case of Horn clauses.

Definition 1. *A* west-ordering *is a well-founded ordering on $T(\mathcal{F})$ that fulfils the subterm property and that is total on $T(\mathcal{F})$ (it is called* west *after well-founded, subterm and total).*

Lemma 7. *[BGNR99] Let R be a ground TRS such that for all rules $l \to r$ in R the term r is irreducible by R. Then R is terminating.*

For a given a west ordering $\succ$, the inference system $\mathcal{J}$ for Horn clauses with equality is (selected equations are written underlined):

paramodulation right:

$$\frac{\to l \simeq r \mid T_1 \qquad \to s \simeq t \mid T_2}{\to s[r]_p \simeq t \mid s|_p = l \wedge T_1 \wedge T_2}$$

if $s|_p \notin \mathcal{X}$ and $l\sigma \succ r\sigma$ for some ground substitution σ which is a solution of $s|_p = l \wedge T_1 \wedge T_2$.

paramodulation left:

$$\frac{\rightarrow l \simeq r \mid T_1 \qquad \Gamma, \underline{s \simeq t} \rightarrow \Delta \ \mid T_2}{\Gamma, s[r]_p \simeq t \rightarrow \Delta \mid s|_p = l \wedge T_1 \wedge T_2}$$

if $s|_p \notin \mathcal{X}$ and $l\sigma \succ r\sigma$ for some ground substitution σ which is a solution of $s|_p = l \wedge T_1 \wedge T_2$.

equality resolution:

$$\frac{\Gamma, \underline{s \simeq t} \rightarrow \Delta \mid T}{\Gamma \rightarrow \Delta \mid s = t \wedge T}$$

Theorem 2. *The inference system $\mathcal{J}$ with equality constraint inheritance is refutationally complete for first-order Horn clauses.*

The proof of theorem 2 is given in appendix B. Now we can use our result of theorem 1 for proving the completeness of a modification $\mathcal{JA}$ of the inference system $\mathcal{J}$. In $\mathcal{JA}$ any strategy selecting a single (positive or negative) equation in each clause is allowed. Note that this new result is not an immediate consequence of theorem 1. The reason is that for right and left paramodulation there is an ordering restriction $l\sigma \succ r\sigma$, and, for explanatory reasons, we did not consider this kind of restrictions in the definition of $\mathcal{N}$. But in the proof transformation from $\mathcal{N}$ to $\mathcal{A}$ one uses a given substitution σ that satisfies all the constraints. This also holds in the case of a proof tree by $\mathcal{J}$ and, moreover, this σ satisfies all the required restrictions $l\sigma \succ r\sigma$ appearing in it. Hence the transformation process works exactly in the same way.

Theorem 3. *The inference system $\mathcal{JA}$ with equality constraint inheritance is refutationally complete for first-order Horn clauses.*

5 Conclusions

We have shown that if a paramodulation-based inference system is complete with a concrete strategy with eager selection of negative equations and, moreover, it is compatible with equality constraint inheritance, then it is complete with arbitrary selection strategies.

Therefore we have generalised the result in [Lyn97] about refutation completeness of arbitrary selection strategies for superposition. Moreover, the generality of our proof transformation method allows us to obtain directly the completeness of arbitrary selection strategies for other inference systems. We have shown that the results in [BGNR99] for paramodulation with non-monotonic orderings are compatible with equality constraint inheritance, thus further restricting the search space and allowing arbitrary selection strategies.

We have also generalised, in a sense, the result in [dN96] for Horn clauses *without* equality, about completeness of resolution with an arbitrary selection of one single literal in each clause.

In [BG94] standard methods for proving compatibility with redundancy elimination techniques are given, by which, roughly, a clause is redundant if it follows from smaller clauses. These notions are not applicable to our proof transformation technique. But this is not surprising, since by these standard techniques all tautologies are redundant, which is not the case here. Some kind of tautologies have to be kept in order to preserve completeness in the case of arbitrary selection strategies, as shown by the following counter example from [Lyn97].

Example 2. Suppose we have

1. $\rightarrow P(c, b, b)$
2. $P(c, c, b), P(c, b, c) \rightarrow b \simeq c$
3. $P(x, y, y) \rightarrow P(x, y, x)$
4. $P(x, y, y) \rightarrow P(x, x, y)$
5. $P(c, c, c) \rightarrow$

and assume an ordering such that $b \succ c$. Now, assume we are in the case of an inference system applying superposition, with a selection rule such that the positive literal is selected in each program clause. The conclusion of every inference in this set of clauses is identical to an existing clause or is a tautology of the form $A, \Gamma \rightarrow A$. However, the set of clauses is unsatisfiable. And, moreover, it can be easily seen that the empty clause is generated if tautologies of the form $A, \Gamma \rightarrow A$ are kept.

5.1 Future work

Our result could be extended in several directions, the most interesting ones being:

- Strategies with non-eager selection of negative equations. We think our proof transformation technique could be adapted to cover also strategies with non-eager selection of negative equations, i.e., proving that if there exists *any* (not necessarily eager negative) complete selection strategy selecting a single literal, then arbitrary selection strategies are complete. A possible way to go could be to first transform such proofs into proofs with eager selection of negative equations.
- Answer computation. In this paper we have focussed on refutation completeness, but we believe that our techniques easily extend to proving the completeness for *answer computation*, thus obtaining similar results for this purpose as the ones of [Lyn97] for total reduction orderings and superposition. The key idea is that solutions σ are preserved during our transformation process.
- Non-equality constraints. Very often it is the case that an inference system works not only with equality constraints but also with other kind of constraints such as, for example, ordering constraints. But, as seen in section 4, the whole transformation process can be done in the same way if there exists

some substitution σ that satisfies all the constraints. Note that this is a reasonable assumption for many inference systems since, in our transformation, the equations involved in the inferences of the proof by $\mathcal{A}$ are the same as in the proof by $\mathcal{N}$. It may be interesting to study with which other kind of constraints this transformation method preserves completeness.

– General clauses. Our result is for first-order Horn clauses with equality. We think that it would not be difficult to adapt our proof transformation method to the case of general clauses, provided no *factoring* inferences occur in the proofs. With factoring, incompleteness already appears in the propositional case, as shown by the following counter example from [dN96].

Example 3. Suppose we have

1. $\rightarrow p, q$
2. $\rightarrow q, r$
3. $\rightarrow r, p$
4. $p, q \rightarrow$
5. $q, r \rightarrow$
6. $r, p \rightarrow$

This set of clauses is unsatisfiable, and factoring (in this propositional case, elimination of repeated occurrences of positive literals) is needed for obtaining the empty clause. Now, suppose we choose the following arbitrary selection:

1. $\rightarrow \underline{p}, q$
2. $\rightarrow \underline{q}, r$
3. $\rightarrow \underline{r}, p$
4. $\underline{p}, q \rightarrow$
5. $\underline{q}, r \rightarrow$
6. $\underline{r}, p \rightarrow$

By applying resolution involving only the selected literals we only obtain tautologies of the form $A \rightarrow A$, and after resolving with/on them we get clauses of the initial set. Therefore this is a counter example to the completeness of arbitrary strategies when factoring is required, even if all tautologies are kept.

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A Proofs of fusion and separation lemmas

Proof. We will give a proof of the fusion lemma that uses the separation lemma, and vice versa. Altogether consists in a combined proof by induction.

1. (Proof of the fusion lemma, assuming the separation lemma is true for $\mathcal{N}$-proofs with a number of steps less than or equal to $steps(P_1) + steps(P_2)$)

If Γ_1 and Γ_2 are empty, then the result is trivial. Otherwise, we analyse which equations are selected in the clause $\Gamma_1, \Gamma_2 \to \Delta$. Wlog. suppose $\mathcal{N}$ selects an equation e in Γ_1. We take $\Gamma_1 = \Gamma_1', e$. We apply now the separation lemma to P_1 and then we have two antecedent elimination $\mathcal{N}$-proofs P_{11} and P_{12} of Γ_1' and e respectively. Moreover, $steps(P_1) = steps(P_{11}) + steps(P_{12})$ and every non-rightmost leaf of P_{11} or of P_{12} is a non-rightmost leaf of P_1. Now, we consider different cases depending on which kind of $\mathcal{N}$-inference is the rightmost step of P_{12}.

1a. If it is an equality resolution then it is of the form:

$$\frac{\underline{e} \to \Delta_e}{\to \Delta_e}$$

It does not matter for us which is Δ_e. Observe that in this case $steps(P_{12}) = 1$, and hence $steps(P_{11}) + steps(P_2) + 1 = steps(P_1) + steps(P_2)$. Then we can apply induction hypothesis to P_{11} and P_2 to obtain an antecedent elimination $\mathcal{N}$-proof P' of $\to \Delta$ s.t. its rightmost leaf is $\Gamma_1', \Gamma_2 \to \Delta$. Moreover, $steps(P') = steps(P_{11}) + steps(P_2)$, and every non-rightmost leaf of P' is a non-rightmost leaf of P_{11} or of P_2, and then of P_1 or of P_2. Now we create P by inserting in the rightmost leaf of P' the $\mathcal{N}$-inference:

$$\frac{\Gamma_1', \underline{e}, \Gamma_2 \to \Delta}{\Gamma_1', \Gamma_2 \to \Delta}$$

And then P satisfies the desired conditions.

1b. If the rightmost step of P_{12} is a paramodulation left inference, then it is of the form:

$$\frac{\to \Delta_3 \qquad \underline{e} \to \Delta_e}{e' \to \Delta_e}$$

Let P_{123} be the subproof of P_{12} rooted by $\to \Delta_3$. And let P_{12}' be like P_{12} but where P_{123} and the rightmost leaf are removed. The rightmost leaf of P_{12}' is $e' \to \Delta_e$, and $steps(P_{12}) = steps(P_{123}) + steps(P_{12}') + 1$. Every non-rightmost leaf of P_{12}' is a non-rightmost leaf of P_{12}, and then of P_1. We have $steps(P_{11}) + steps(P_{12}') + steps(P_2) < steps(P_1) + steps(P_2)$. Then we can apply, two times, induction hypothesis to P_{11}, P_{12}' and P_2 to obtain an antecedent elimination $\mathcal{N}$-proof P' of $\to \Delta$ s.t. its rightmost leaf is $\Gamma_1', e', \Gamma_2 \to \Delta$. Moreover, $steps(P') = steps(P_{11}) + steps(P_{12}') + steps(P_2)$, and every non-rightmost leaf of P' is a non-rightmost leaf of P_{11} or of P_{12}' or of P_2, and then of P_1 or of P_2. Now we create P by inserting in the rightmost leaf of P' the $\mathcal{N}$-inference

$$\frac{\rightarrow \Delta_3 \qquad \Gamma_1', \underline{e}, \Gamma_2 \rightarrow \Delta}{\Gamma_1', e', \Gamma_2 \rightarrow \Delta}$$

and inserting P_{123} above $\rightarrow \Delta_3$. Then, P satisfies then the desired conditions.

2. (Proof of the separation lemma, assuming the fusion lemma is true for pairs of $\mathcal{N}$-proofs s.t. the sum of their steps is less than $steps(P)$)

If Γ_1 or Γ_2 is empty then the result is trivial. Otherwise suppose wlog. that the selected equation e in $\Gamma_1, \Gamma_2 \rightarrow \Delta$ is in Γ_1. We take $\Gamma_1 = \Gamma_1', e$. Then there are two cases to be considered, depending on what kind of inference is the rightmost step of P.

2a. If it is an equality resolution, then it is of the form:

$$\frac{\Gamma_1', \underline{e}, \Gamma_2 \rightarrow \Delta}{\Gamma_1', \Gamma_2 \rightarrow \Delta}$$

Let P' be like P but where this step is removed. We have $steps(P') + 1 = steps(P)$. Therefore we can apply induction hypothesis to P' and, then, there exist antecedent elimination proofs P_1' and P_2 of Γ_1' and Γ_2 respectively. Moreover, their non-rightmost leaves are non-rightmost leaves of P', and then of P, and $steps(P_1') + steps(P_2) = steps(P')$. Consider now the following $\mathcal{N}$-inference:

$$\frac{\underline{e} \rightarrow \Delta}{\rightarrow \Delta}$$

It can be seen as an $\mathcal{N}$-proof with only one step. It is an antecedent elimination proof of e, that we name P_e. Now observe that, since Γ_2 is not empty, we have $steps(P_2) \geq 1$, and hence $steps(P_1') + steps(P_e) = steps(P_1) < steps(P_1) + steps(P_2) = steps(P)$. Therefore we can apply, inductively, the fusion lemma to P_1' and P_e, obtaining an antecedent elimination $\mathcal{N}$-proof P_1 of Γ_1', e, that is Γ_1. Every non-rightmost leaf of P_1 is a non-rightmost leaf of P_1' (since P_e does not contain non-rightmost leaves), and hence, of P; and $steps(P_1) = steps(P_1') + steps(P_e) = steps(P_1') + 1$. Therefore $steps(P) = steps(P_1) + steps(P_2)$.

2b. If the rightmost step of P is a paramodulation left inference, then, it is of the form:

$$\frac{\rightarrow \Delta_3 \qquad \Gamma_1', \underline{e}, \Gamma_2 \rightarrow \Delta}{\Gamma_1', e', \Gamma_2 \rightarrow \Delta}$$

Let P_3 be the subproof of P rooted by $\rightarrow \Delta_3$. Let P' be like P but where P_3 and the rightmost leaf are removed. The root of P' is $\Gamma_1', e', \Gamma_2 \rightarrow \Delta$, and $steps(P) = steps(P') + steps(P_3) + 1$. We can apply induction hypothesis to P', two times, and obtain three antecedent elimination $\mathcal{N}$-proofs P_1', $P_{e'}$ and P_2 of Γ_1', e' and Γ_2 respectively, s.t. all their non-rightmost leaves are non-rightmost leaves of P', and then of P. Moreover, $steps(P') = steps(P_1') + steps(P_{e'}) + steps(P_2)$. Now we create P_e by inserting in the rightmost leaf of $P_{e'}$ the inference

$$\frac{\rightarrow \Delta_3 \qquad \underline{e} \rightarrow \Delta}{e' \rightarrow \Delta}$$

and inserting P_3 above $\rightarrow \Delta_3$, and hence $steps(P_e) = steps(P_{e'}) + steps(P_3) + 1$. Observe that every non-rightmost leaf of P_e is a non-rightmost leaf of P, since it is a non-rightmost leaf of $P_{e'}$ or a leaf of P_3. We have $steps(P) = steps(P') + steps(P_3) + 1 = steps(P_1') + steps(P_{e'}) + steps(P_2) + steps(P_3) + 1 = steps(P_1') + steps(P_e) + steps(P_2)$. Now, since Γ_2 is not empty, we have $steps(P_2) \geq 1$, and hence $steps(P_1') + steps(P_e) < steps(P)$. Therefore, we can apply the fusion lemma to P_1' and P_e, obtaining P_1, an antecedent elimination $\mathcal{N}$-proof of Γ_1', e, that is Γ_1. Moreover, their non-rightmost leaves are non-rightmost leaves of P_1' or of P_e, and hence of P. Furthermore $steps(P_1) = steps(P_1') + steps(P_e)$. Therefore $steps(P) = steps(P_1) + steps(P_2)$. □

Proof. (of general fusion lemma) Let P_{21} be the antecedent elimination $\mathcal{N}$-proof on the rightmost path of P_2. Let P_2' be like P_2 but where P_{21} is removed. Note that the rightmost leaf of P_2' is $\rightarrow \Delta_2$, and $steps(P_2) = steps(P_2') + steps(P_{21})$. By the fusion lemma, since P_1 and P_{21} are antecedent elimination proofs, there exists P' s.t. it is antededent elimination of $\Gamma_1 \cup \Gamma_2$, and $steps(P') = steps(P_1) + steps(P_{21})$, and all the non-rightmost leaves of P' are non-rightmost leaves of P_1 or of P_{21}, and hence of P_1 or of P_2. Moreover, the root of P' can be chosen to be $\rightarrow \Delta_2$, maintaining these conditions. Then the rightmost leaf of P' is $\Gamma_1, \Gamma_2 \rightarrow \Delta_2$. Now, let P be the $\mathcal{N}$-proof formed by inserting P' on the rightmost leaf of P_2'. Then P satisfies all the conditions we are looking for. □

Proof. (of general separation lemma) Let P_{21} be the subtree in the rightmost path of P that is antecedent elimination of Γ_1, Γ_2. We have that P_{21} is a proof of $\rightarrow \Delta'$. Let P_2' be like P but removing P_{21} from it. We have that the rightmost leaf of P_2' is $\rightarrow \Delta'$ and $steps(P) = steps(P_2') + steps(P_{21})$. Now we apply the separation lemma to P_{21} and we obtain an antecedent elimination proof P_1 of Γ_1, and an antecedent elimination proof P_{21}' of Γ_2. By lemma 3, we can assume that the rightmost leaf of P_{21}' is $\Gamma_2 \rightarrow \Delta'$. Now we define P_2 to be like P_2' but inserting P_{21}' in its rightmost leaf. Then P_1 and P_2 satisfy the required conditions. □

B Proof of theorem 2

Definition 2. *Given an ordering $\succ$, we define $\succ_{mul}$ to be the smallest ordering on multisets such that:*

$$M \cup \{s\} \succ_{mul} N \cup \{t_1, \ldots, t_n\} \text{ if } M = N \text{ and } s \succ t_i \text{ for all } i \in 1 \ldots n.$$

If $\succ$ is well founded and total on S, so is $\succ_{mul}$ on finite multisets over S [DM79].

This will be used to lift orderings $\succ$ on terms to orderings on equations and clauses:

Definition 3. *Let C be a ground clause, and let $emul(s \simeq t)$ be $\{s,t\}$ if $s \simeq t$ is a positive equation in C, and $\{s,s,t,t\}$ if it is negative. Then, if $\succ$ is an ordering, we define the ordering $\succ_e$ on (occurrences of) ground equations in a clause by $e \succ_e e'$ if $emul(e) \succ_{mul} emul(e')$. Similarly, $\succ_c$ on ground clauses is defined $C \succ_c D$ if $mse(C) \ (\succ_{mul})_{mul} \ mse(D)$, where $mse(C)$ is the multiset of all $emul(e)$ for occurrences e of equations in C.*

Definition 4. *Let S be a set of clauses. An instance C of the form $\rightarrow l \simeq r$ of a clause in S* generates *the rule $l \rightarrow r$ if*

1. *$l \succ r$ and*
2. *l and r are irreducible by R_C.*

where R_C is the set of rules generated by all instances D of clauses in S such that $C \succ_c D$. We denote by R_S the set of rules generated by all ground instances of S.

Lemma 8. *Let S be a set of clauses. Then the ground TRS R_S is convergent.*

Proof. For termination, by Lemma 7 it suffices to show that for every rule $l \rightarrow r$ in R_S the term r is irreducible by R_S. By construction, if an instance C generates $l \rightarrow r$, the term r is irreducible by R_C. Since $l \succ r$, and $\succ$ fulfils the subterm property, clearly $l \rightarrow r$ itself does not reduce r either. Finally, if $l' \rightarrow r'$ is generated by an instance D with $D \succ_c C$ then, by definition of $\succ_c$, we must have $l' \succ l$ and hence l' cannot be a subterm of r either. This proves termination.

For similar reasons l is irreducible by $R_S \setminus \{l \rightarrow r\}$. This means that $\rightarrow_{R_S}$ is locally confluent and hence, since it is terminating, confluent. □

Definition 5. *Let S be a set of clauses. By $\succ_R$ we denote the ordering $\rightarrow^+_{R_S}$.*

Lemma 9. *Let S be a set of clauses. Then $\succ_R$ is a reduction ordering.*

Proof. Since R_S is a terminating TRS, $\rightarrow^+_{R_S}$ is a reduction ordering. [DJ90]. □

Proof. (of theorem 2) Let S_0 be a set of —unconstrained— Horn clauses, and let S be the closure of S_0 under $\mathcal{J}$ with equality constraint inheritance. We show that R^*_S is a model for S whenever $\Box \notin S$.

We first prove that $R^*_S \models irred_{R_S}(S)$, where R^*_S is the equality Herbrand interpretation defined by the congruence $\overset{*}{\leftrightarrow}_{R_S}$ and $irred_{R_S}(S)$ is the set of ground instances $C\sigma$ of clauses $C \mid T$ in S such that $\sigma \models T$ and σ is irreducible by R_S (σ is *irreducible* by R_S if $x\sigma$ is irreducible for all $x \in Dom(\sigma)$; furthermore, we denote by $nf_{R_S}(\sigma)$ the substitution σ' such that $x\sigma' = nf_{R_S}(x\sigma)$ for all $x \in Dom(\sigma)$).

We proceed by induction on $(\succ_R)_c$, that is, we derive a contradiction from the existence of a minimal (wrt.$(\succ_R)_c$) ground instance $C\sigma \in irred_{R_S}(S)$ for some $C \mid T \in S$, where $\sigma \models T$, such that $R^*_S \not\models C\sigma$. In the following, we (ambiguously) write $\succ_R$ for terms, equations and clauses instead of $\succ_R$, $(\succ_R)_e$ and $(\succ_R)_c$ respectively:

1. If C is of the form $\rightarrow s \simeq t$ then we assume, wlog., that $C\sigma$ is of the form $\rightarrow s\sigma \simeq t\sigma$ with $s\sigma \succ t\sigma$. Since $R_S^* \not\models C\sigma$, we have that $s\sigma \rightarrow t\sigma \notin R_S$, i.e. $C\sigma$ has not generated the rule $s\sigma \rightarrow t\sigma$ due to one of the following reasons: either $s\sigma$ or $t\sigma$ are reducible by R_C. Let us consider that $s\sigma$ is reducible (the proof for $t\sigma$ is analogous). Assume $s\sigma$ is reducible by some rule $l\sigma \rightarrow r\sigma \in R_S$, and $l\sigma \rightarrow r\sigma$ has been generated by an instance $C'\sigma$ of some clause $C' \mid T'$ in S, where C' is of the form $\rightarrow l \simeq r$ with $l\sigma \succ r\sigma$. Now we have $s\sigma|_p \equiv l\sigma$ and, since σ is irreducible by R_S, the only possibility is now that p is a non-variable position of s. (Note that, since we assume that there are no name clashes between the—different—variables of C and C', we can consider that the instances of C and of C' under consideration are both by the same ground σ.) Then there exists an inference by paramodulation right:

$$\frac{\rightarrow l \simeq r \ \mid T' \qquad \rightarrow s \simeq t \ \mid T}{\rightarrow s[r]_p \simeq t \ \mid s|_p = l \wedge T' \wedge T}$$

whose conclusion D has an instance $D\sigma$ where $\sigma \models s|_p = l \wedge l > r \wedge T \wedge T'$ such that $C\sigma \succ_R D\sigma$ and where $R_S^* \not\models D\sigma$. Furthermore, $D\sigma \in irred_{R_S}(S)$: indeed $x\sigma$ is irreducible by R_S for all variables $x \in vars(D)$. This is clearly the case if $x \in vars(C)$. For $x \in vars(C')$, there are two possibilities: if $x \equiv l$ then $x \notin vars(D)$ since $l\sigma \succ r\sigma$ and $\succ \supset \triangleleft$; if $x \not\equiv l$ then $x\sigma$ is irreducible wrt. $R_{C'}$ by construction of R_S, and also wrt. R_S, since for all rules $l' \leftarrow r' \in R_S \setminus R_{C'}$ we have $l' \succeq l\sigma \succ x\sigma$ and hence such rules cannot reduce $x\sigma$. Altogether, this contradicts the minimality of $C\sigma$.

2. If C is of the form $\Gamma, \underline{s \simeq t} \rightarrow \Delta$, from the fact that $R_S^* \not\models C\sigma$ it necessarily follows that $s\sigma \simeq t\sigma \in R_S^*$ and, since R_S is convergent, $s\sigma$ and $t\sigma$ must have the same normal form wrt. R_S, i.e. there must exist a term v such that $s\sigma \ \xrightarrow{*}_{R_S} v \ \xleftarrow{*}_{R_S} \ t\sigma$. The following cases have to be considered:

2a. $s\sigma \equiv t\sigma$. In this case there exists an inference by equality resolution:

$$\frac{\Gamma, \underline{s \simeq t} \rightarrow \Delta \ \mid T}{\Gamma \rightarrow \Delta \ \mid s = t \wedge T}$$

whose conclusion D has an instance $D\sigma$ with the same but less terms than $C\sigma$, and hence $C\sigma \succ_R D\sigma$. Since $R_S^* \not\models C\sigma$ we also have $R_S^* \not\models D\sigma$. This contradicts the minimality of $C\sigma$.

2b. $s\sigma \not\equiv t\sigma$ and $s\sigma \ \xrightarrow{+}_{R_S} \ v \ \xleftarrow{*}_{R_S} \ t\sigma$. In this case $s\sigma$ is reducible at some non-variable position p by some rule $l\sigma \rightarrow r\sigma$, which has been generated by an instance $C'\sigma$ of some clause $C' \mid T'$ in S, where C' is of the form $\rightarrow l \simeq r$ with $l\sigma \succ r\sigma$. Then there exists an inference by paramodulation left:

$$\frac{\rightarrow l \simeq r \ \mid T' \qquad \Gamma, \underline{s \simeq t} \rightarrow \Delta \ \mid T}{\Gamma, s[r]_p \simeq t \rightarrow \Delta \mid s|_p = l \wedge T' \wedge T}$$

whose conclusion D has an instance $D\sigma$ such that $C\sigma \succ_R D\sigma$ and $R_S^* \not\models D\sigma$. Furthermore, $D\sigma \in irred_{R_S}(S)$: for all variables $x \in vars(D)$ such that $x \in$

$vars(C)$ we have that $x\sigma$ is irreducible by R_S; for all variables $x \in vars(C')$ we have $x \equiv l$ or $x \not\equiv l$, and the two situations are solved in the same way as before. Altogether, this contradicts the minimality of $C\sigma$.

2c. $s\sigma \not\equiv t\sigma$ and $s\sigma \overset{+}{\leftarrow}_{R_S} t\sigma$. This case is analogous to the previous one.

Once we have $R_S^* \models irred_{R_S}(S)$, then also $R_S^* \models irred_{R_S}(S_0)$ since $S \supseteq S_0$. But then $R_S^* \models S_0$ as well, since for each ground instance $C\sigma$ of a clause in S_0, if $\sigma' = nf_{R_S}(\sigma)$ then $C\sigma'$ is an existing instance of a clause in $irred_{R_S}(S_0)$ as clauses in S_0 have no constraints, and then $R_S^* \models C\sigma'$, which clearly implies $R_S^* \models C\sigma$. Finally, since $S_0 \models S$, from $R_S^* \models S_0$ we obtain $R_S^* \models S$. □