Well-foundedness is sufficient for completeness of Ordered Paramodulation

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Abstract. For many years all known completeness results for Knuth-Bendix completion and ordered paramodulation required the term ordering $\succ$ to be well-founded, monotonic and total(izable) on ground terms. Then, it was shown that well-foundedness and the subterm property were enough for ensuring completeness of ordered paramodulation.
Here we show that the subterm property is not necessary either. By using a new restricted form of rewriting we obtain a completeness proof of ordered paramodulation for Horn clauses with equality where well-foundedness of the ordering suffices. Apart from the fundamental interest of this result, some potential applications motivating the interest of dropping the subterm property are given.

1 Introduction

Knuth-Bendix-like completion techniques and their extensions to ordered paramodulation for first-order clauses are among the most successful methods for automated deduction with equality [BG98,NR01]. For many years all known completeness results for Knuth-Bendix completion and ordered paramodulation required the term ordering $\succ$ to be well-founded, monotonic and total (or extendable to a total ordering) on ground terms [HR91,BDH86,BD94,BG94], until in [BGNR99], the monotonicity requirement was dropped and well-foundedness and the subterm property were shown to be sufficient (note that any such ordering can be totalized without loosing these two properties).

After this, the fundamental question arises whether more requirements can be dropped. In this paper we prove that for ordered paramodulation on Horn clauses with equality the subterm property can be dropped as well. Therefore only well-foundedness is required. These results are a new important step in the theory

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of paramodulation, which somehow shows the power of ordered paramodulation regardless of the properties of the ordering which is used and leaves as the last question whether even well-foundedness is necessary.

Apart from its theoretical value, there are at least two other potential applications of these results. The first one is in deduction modulo built-in equational theories E, since for many E, finding E-*compatible* orderings fulfilling the required properties is extremely complex (for example for the associative and commutative (AC) case) or even impossible (for instance in presence of idempotence axioms [BGNR99]). Requiring only well-foundedness allows us to consider any theory E, since there is always a well-founded E-compatible ordering (note that, as any set, the set of E-congruence classes admits a well-founded ordering).

The second potential application comes from the fact that, sometimes, a goal-oriented (ordered) paramodulation proof can only be obtained if the ordering contradicts the subterm property. Let us illustrate it with a simple example.

Example 1. Consider the following set of equations:

$$\begin{array}{lr@{\;}l} & y^{log_y x} & \simeq x \\ & log_y y^x & \simeq x \\ & x^{y+z} & \simeq x^y \cdot x^z \\ \text{and the goal} \qquad & log_a b + log_a c & \simeq log_a(b \cdot c). \end{array}$$

With the following well-founded ordering

$$\begin{array}{rll} y^{log_y x} & \succ x & \\ x & \succ log_y y^x & \text{if } x \text{ is not headed by } log \\ log_y y^x & \succ x & \text{if } x \text{ is headed by } log \\ x^{y+z} & \succ x^y \cdot x^z & \end{array}$$

which contradicts the subterm property, we have a goal-oriented proof by ordered paramodulation (note that some variable renamings have been applied).

$$\frac{\boxed{x} \simeq log_y y^x \qquad \boxed{log_a b + log_a c} \not\simeq log_a(b \cdot c)}{log_y \boxed{y^{log_a b + log_a c}} \not\simeq log_a(b \cdot c)}$$

$$\frac{\boxed{x_1^{y_1+z_1}} \simeq x_1^{y_1} \cdot x_1^{z_1} \qquad log_y \boxed{y^{log_a b + log_a c}} \not\simeq log_a(b \cdot c)}{log_y(\boxed{y^{log_a b}} \cdot y^{log_a c}) \not\simeq log_a(b \cdot c)}$$

$$\frac{\boxed{y_1^{log_{y_1} x_1}} \simeq x_1 \qquad log_y(\boxed{y^{log_a b}} \cdot y^{log_a c}) \not\simeq log_a(b \cdot c)}{log_a(b \cdot \boxed{a^{log_a c}}) \not\simeq log_a(b \cdot c)}$$

$$\frac{\boxed{y_1^{log_{y_1} x_1}} \simeq x_1 \qquad log_a(b \cdot \boxed{a^{log_a c}}) \not\simeq log_a(b \cdot c)}{log_a(b \cdot c) \not\simeq log_a(b \cdot c)}$$

$$\frac{log_a(b \cdot c) \not\simeq log_a(b \cdot c)}{\square}$$

Note that there is no such fully goal-oriented proof with respect to a well-founded ordering including the subterm relation, because no paramodulation step could be applied on the goal: for the first two equations, due to the subterm property, only the term on the left can be used and it does not unify with any subterm of the goal; for the last equation neither side overlaps with the goal. □

In this paper we introduce techniques for dropping the subterm (as well as the monotonicity) requirement. Our technique (given in Sections 4 and 5) is a new variant of the model generation technique, where, surprisingly, the ground rewrite system R that defines the model may not be terminating. For this reason, we define a *new* restricted form of rewriting, called *blocked rewriting* ($\mathcal{B}$-rewriting for short), which is terminating by definition.

$\mathcal{B}$-rewriting works on *abstracted terms*, denoted by $t \cdot \gamma$, which are terms split into two parts: the *abstraction (or term) part* t and the *substitution part* γ, such that $t\gamma$ is a term (this formalism corresponds to the *closures* defined for clauses in [BGLS95]). No rewrite steps can be applied to the γ part and when we rewrite $t \cdot \gamma$ with a rule $l \to r$ at a non-variable position p in t we obtain $t[x]_p \cdot (\gamma \cup \{x \mapsto r\})$ for some fresh variable x. It is obvious that this rewrite relation terminates since each step strictly reduces the size of the non-variable part t.

Example 2. Consider the set of equations E containing only

$$g(a) \simeq a$$

and an ordering such that $a \succ g(a)$ and $g^k(a) \succ g^j(a)$ with $k \geq 2$ and $k > j \geq 0$. In this ordering the minimal term is $g(a)$.

If we close this set E with respect to the ordered paramodulation rule, we obtain a set of equations of the form

$$g^k(a) \simeq g^j(a) \text{ with } k > j \geq 0.$$

which are oriented as

$$\begin{array}{rl} a \to g(a) & \\ g^k(a) \to g^j(a) & \text{for all } k \geq 2 \text{ and } k > j \geq 0 \end{array}$$

Then the following set R containing the two rules

$$\begin{array}{r} a \to g(a) \\ g(g(a)) \to g(a) \end{array}$$

is a confluent (although overlapping) rewrite system for $\mathcal{B}$-rewriting, and furthermore all equations in the closure can be proved by $\mathcal{B}$-rewriting using R. In particular, all $g^k(a) \simeq g^j(a)$ can be proved by joining both sides into the same abstracted term $x \cdot \{x \mapsto g(a)\}$. Therefore, for every consequence of E there is a $\mathcal{B}$-rewrite proof using R. □

In our completeness proof a (possibly infinite) rewrite system like the R above is generated. Proving confluence of R is essential but, since the system may be overlapping, this property does not follow easily. Once we have proved it, we can show that there is a $\mathcal{B}$-rewrite proof with R for every consequence of the unit positive equations. Finally, this is used to prove the completeness of ordered paramodulation for first-order Horn clauses with eager *selection* of negative literals.

Besides the proof of confluence (which is the hardest and more technical part), the core of the proof is rather simple and similar to the one in [BGNR99], since it still uses induction on the $\mathcal{B}$-rewrite relation with R (which is well-founded) for the completeness result.

The paper is structured as follows. The next section provides the basic definitions. In Section 3 $\mathcal{B}$-rewriting is introduced and its main properties stated. Section 4 is devoted to the equational case. There, the original part of our proof technique is presented and the difficulties of proving the confluence of the generated rewrite system that defines the model are outlined. In section 5, completeness for Horn clauses is proved using the results obtained for the equational case. Finally in section 6 conclusions and future developments are discussed. All proofs can be found in [BR02].

2 Basic notions

We use the standard definitions of [DJ90]. $T(\mathcal{F},\mathcal{X})$ $(T(\mathcal{F}))$ is the set of (ground) terms over a set of symbols $\mathcal{F}$ and a denumerable set of variables $\mathcal{X}$ (over $\mathcal{F}$). By $vars(t)$ we denote the set of variables occurring in a term t. The size of a term t is denoted by $|t|$ and its size without counting the variables by $|t|_v$.

The subterm of t at *position* p is denoted by $t|_p$, the result of replacing $t|_p$ by s in t is denoted $t[s]_p$, and syntactic equality of terms is denoted by $\equiv$. Let t be a term in $T(\mathcal{F},\mathcal{X})$ and M a set of terms in $T(\mathcal{F},\mathcal{X})$. We denote by $P_M(t)$ the set of positions p of t s.t. there is some prefix p' of p s.t. $t|_{p'} \in M$, i.e. if $t|_p \in M$ then $P_M(t)$ includes p and all positions below p. By $P_{\mathcal{X}}(t)$ we denote the set of variable positions of t, i.e. $P_{vars(t)}(t)$.

If $\rightarrow$ is a binary relation, then $\leftarrow$ is its inverse, $\leftrightarrow$ is its symmetric closure, $\xrightarrow{+}$ is its transitive closure and $\xrightarrow{*}$ is its reflexive-transitive closure. We write $s \xrightarrow{!} t$ if $s \xrightarrow{*} t$ and there is no t' such that $t \rightarrow t'$. Then t is called *irreducible* and a *normal form* of s (wrt. $\rightarrow$). If a term s has exactly one normal form then it is denoted by $s!_{\rightarrow}$. The relation $\rightarrow$ is *well-founded* or *terminating* if there exists no infinite sequence $s_1 \rightarrow s_2 \rightarrow \dots$ and it is *confluent* or *Church-Rosser* if the relation $\leftarrow^* \circ \rightarrow^*$ is contained in $\rightarrow^* \circ \leftarrow^*$. It is *locally confluent* if $\leftarrow \circ \rightarrow \; \subseteq \; \rightarrow^* \circ \leftarrow^*$. By Newman's lemma, terminating locally-confluent relations are confluent. A relation $\rightarrow$ on terms is *monotonic* if $s \rightarrow t$ implies $u[s]_p \rightarrow u[t]_p$ for all terms s, t and u and positions p. A *congruence* is a reflexive, symmetric, transitive and monotonic relation on terms.

An *equation* is a multiset of terms $\{s, t\}$, denoted $s \simeq t$ or, equivalently, $t \simeq s$. By $gnd(s \simeq t)$ we denote the set of ground instances of $s \simeq t$. Let E be a set of equations. By $gnd(E)$ we denote the set of ground instances of all equations in E.

A first-order clause is a pair of finite multisets of equations Γ (the *antecedent*) and Δ (the *succedent*), denoted by $\Gamma \rightarrow \Delta$. Equations in Γ are called *negative literals* and equations in Δ are called *positive literals*. A Horn clause is a clause with at most one positive literal. The empty clause $\square$ is a clause where both Γ and Δ are empty.

A *rewrite rule* is an ordered pair of terms (s, t), written $s \rightarrow t$, and a set of rewrite rules R is a *term rewrite system* (TRS). By $Left(R)$ and $Right(R)$ we denote, respectively, the set of left-hand sides of rules in R and the set of right-hand sides of rules in R. The rewrite relation with R on $T(\mathcal{F}, \mathcal{X})$, denoted $\rightarrow_R$, is the smallest monotonic relation such that $l\sigma \rightarrow_R r\sigma$ for all $l \rightarrow r \in R$ and all substitutions σ, and if $s \rightarrow_R t$ then we say that s *rewrites into* t with R. R is called terminating, confluent, etc. if $\rightarrow_R$ is.

A (strict partial) *ordering* on $T(\mathcal{F}, \mathcal{X})$ is an irreflexive transitive relation $\succ$. It fulfils the *subterm property* if $\succ \supseteq \rhd$, where $\rhd$ denotes the strict subterm ordering.

Definition 1. *A* west-ordering *is a well-founded ordering on $T(\mathcal{F})$ that fulfils the subterm property and that is total on $T(\mathcal{F})$ (it is called* west *after well-founded, subterm and total).*

Every well-founded ordering can be totalized [Wec91], and hence every well-founded ordering satisfying the subterm property can be extended to a west-ordering.

3 $\mathcal{B}$-rewriting

Now we define a special kind of rewrite relation, called *blocked rewriting* ($\mathcal{B}$-rewriting for short), which is always terminating. This relation restricts standard rewriting by mainly forbidding rewrite steps on right-hand sides introduced by previous rewrite steps. Roughly speaking, after every rewrite step the introduced right-hand side becomes blocked.

In order to express this blocking we use *abstracted terms* which have variables in the blocked positions. The blocked part of the term is stored in a substitution.

In what follows we only consider ground term rewrite systems, which is the only case we need. However, all definitions can easily be extended to deal with a non-ground TRS.

Definition 2. *Let R be a TRS. Let γ be a ground substitution and s a linear term with $vars(s) \subseteq Dom(\gamma)$ and $Im(\gamma) \subseteq Right(R)$. Then $s \cdot \gamma$ is called an* abstracted term *wrt. R, and an* abstraction *of $s\gamma$ wrt. R.*

In the following, we write s instead of $s \cdot \gamma$ whenever γ denotes the empty substitution.

Two abstracted terms $s \cdot \gamma$ and $t \cdot \delta$ are equivalent, denoted by $s \cdot \gamma \equiv t \cdot \delta$, if s and t are equal up to renaming of variables and $s\gamma \equiv t\delta$.

An abstracted term $t' \cdot \delta$ is called an argument abstraction *wrt. R of a term $t = f(t_1, \ldots, t_n)$ if $t' = f(t'_1, \ldots, t'_n)$ and every $t'_i \cdot \delta$ is an abstraction of t_i which cannot be further abstracted. We (ambiguously) write $arg\_abs(t, R)$ to denote any argument abstraction of t wrt. R, since all argument abstractions of t wrt. R are equivalent.*

Given an abstracted term $s \cdot \gamma$ wrt. R, the set of *blocked positions* is the set of positions q in $s\gamma$ such that q is below or equal to a variable position p in s (i.e. $p \in P_{\mathcal{X}}(s)$).

Now, we can define $\mathcal{B}$-rewriting as a rewrite relation on abstracted terms.

Definition 3. *Let R be a TRS and $s \cdot \gamma$ an abstracted term wrt. R. Then $s \cdot \gamma \curvearrowright_R s[x]_p \cdot (\gamma \cup \{x \mapsto r\})$ if $s|_p \notin \mathcal{X}$, $x \in \mathcal{X} \setminus vars(s)$, and either $s\gamma|_p \to r \in R$ or $s\gamma|_p \equiv r \in Right(R)$.*

If $s\gamma|_p \equiv r \in Right(R)$ we refer to the rewrite step as an abstraction step.

We will ambiguosly write $s \cdot \gamma \curvearrowright_S t \cdot \delta$ to denote a non-abstraction step using a rule of a subset S of R.

$\mathcal{B}$-rewriting fulfils the following properties:

Lemma 1. *Let R be a TRS. Then $\overset{+}{\curvearrowright}_R$ is a well-founded ordering.*

Lemma 2. *Let R be a TRS. Then $\overset{+}{\curvearrowright}_R$ is compatible with the equivalence on abstracted terms, i.e., if $s' \cdot \gamma' \equiv s \cdot \gamma \overset{+}{\curvearrowright}_R t \cdot \delta \equiv t' \cdot \delta'$ then $s' \cdot \gamma' \overset{+}{\curvearrowright}_R t' \cdot \delta'$.*

Lemma 3. *Let R be a TRS. Then $\overset{+}{\curvearrowright}_R$ is a monotonic relation, in the sense that $s \cdot \gamma \overset{+}{\curvearrowright}_R t \cdot \delta$ implies $C[s]_p \cdot (\rho \cup \gamma) \overset{+}{\curvearrowright}_R C[t]_p \cdot (\rho \cup \delta)$, where $C \cdot \rho$ is an abstracted context (i.e. an abstracted term with a hole, also denoted by $C[\,]_p \cdot \rho$).*

We must note that what identifies a rewrite step is the rule which is applied and the position where it is applied, regardless of the context. Similarly, a derivation is identified by a sequence of steps. However, we usually refer (ambiguously) to a derivation making the context where it is applied explicit. Then, given a TRS R, we refer to $s \cdot \gamma \overset{*}{\curvearrowright}_R t \cdot \delta$ as a derivation with $\curvearrowright_R$ of $t \cdot \delta$ from $s \cdot \gamma$.

Now we define two restricted versions of $\curvearrowright_R$, denoted by $\curvearrowright_{\underline{R}}$ and $\curvearrowright_{\overline{R}}$, which respectively consist of rewriting (by non-abstraction steps) only above blocked positions and only above non-blocked positions. Recall that the only forbidden steps for $\curvearrowright_R$ are the ones at or under blocked positions. For instance in the abstracted term $f(a, x) \cdot \{x \mapsto g(a)\}$, we can rewrite with $\curvearrowright_{\underline{R}}$ only at position λ and with $\curvearrowright_{\overline{R}}$ only at position 1.

Definition 4. *Let R be a TRS and $s \cdot \gamma$ an abstracted term wrt. R.*

We write $s \cdot \gamma \curvearrowright_{\underline{R}} s[x]_p \cdot (\gamma \cup \{x \mapsto r\})$ if $s \cdot \gamma \curvearrowright_R s[x]_p \cdot (\gamma \cup \{x \mapsto r\})$, $s\gamma|_p \to r \in R$ and $vars(s|_p) \neq \emptyset$.

We write $s \cdot \gamma \curvearrowright_{\overline{R}} s[x]_p \cdot (\gamma \cup \{x \mapsto r\})$ if $s \cdot \gamma \curvearrowright_R s[x]_p \cdot (\gamma \cup \{x \mapsto r\})$, $s\gamma|_p \to r \in R$ and $vars(s|_p) = \emptyset$.

Note that one relation is the complement of the other one if we restrict to non-abstraction steps.

4 Paramodulation with equations

In the following, let $\succ$ be a given well-founded ordering (total) on ground terms.

Definition 5. *The inference rule of (equational)* ordered paramodulation *with respect to* $\succ$ *is:*

$$\frac{l \simeq r \qquad s \simeq t}{(s[r]_p \simeq t)\sigma}$$

where $\sigma = mgu(s|_p, l)$ *is the most general unifier of* $s|_p$ *and* l*, where* l *is maximal in its premise (that is, for some ground substitution* θ*, it holds that* $l\sigma\theta \succ r\sigma\theta$*) and* $s|_p$ *is not a variable or, if* $s|_p$ *is a variable* x*, then* $x \in vars(t)$*,* $p = \lambda$ *and* l *is also a variable.*

Note that this last condition implies that in very restrictive cases we have to perform an inference on a variable position. However, these potential inferences on variable positions are not prolific at all, since both s *and* l *should be variables (with* s *occurring in* t*, which may happen since the ordering does not need to satisfy the subterm property). Hence the only possible inference has* $r\sigma \simeq t$ *as conclusion where* σ *is the variable renaming of* l *by* s*.*

We associate a ground TRS $\vec{E}$ to a set of equations E.

Definition 6. *Let* E *be a set of equations. We define* $\vec{E}$ *as the ground TRS obtained by orienting wrt.* $\succ$ *the ground instances of all equations in* E*. That is,*

$$\vec{E} = \{s \to t \mid s \simeq t \in gnd(E) \wedge s \succ t\}.$$

Example 3. Following Example 2 in the introduction, as said, the closure by ordered paramodulation of the equation $a \simeq g(a)$, consists of all equations of the form $g^k(a) \simeq g^j(a)$ with $k > j \geq 0$. Then, with the considered well-founded ordering, $\vec{E}$ consists of the rules

$$\begin{array}{rl} a \to g(a) & \\ g^k(a) \to g^j(a) & \text{for all } k \geq 2 \text{ and } k > j \geq 0 \end{array}$$

$\square$

In the following we assume E to be a set of equations closed under ordered paramodulation with respect to a well-founded ordering $\succ$. We also assume that the variables of every equation are disjoint (we rename, with fresh variables, the variables occurring in the conclusion of every inference).

Definition 7. *Let* $\succ_{\rhd}$ *be an arbitrary west-ordering on* $T(\mathcal{F})$*, and* $\succ$ *the well-founded ordering used in the paramodulation rule. We define the ordering* $\succ_E$ *on* $\vec{E}$ *as follows: given two rules* $l \to r$ *and* $l' \to r'$ *in* $\vec{E}$*, we have* $l \to r \succ_E l' \to r'$ *if and only if* $l \succ_{\rhd} l'$*, or* $l \equiv l'$ *and* $r \succ r'$*.*

Since $\succ_E$ is well-founded, we can now define, by induction on $\succ_E$, a ground TRS $\mathcal{R}$ *generated* by $\vec{E}$, and such that $\mathcal{R} \subseteq \vec{E}$. The *blockable-terms*, or simply *$\mathcal{B}$-terms*, are all terms that are right-hand sides of rules in $\mathcal{R}$.

Definition 8. *A rule e of the form $s \to t$ in $\vec{E}$* generates *the rule $s \to t$ if*

1. *s is irreducible by $\mathcal{R}^e$ at positions not in $P_{\mathcal{B}^e}(s)$ and*
2. *t is irreducible by $\mathcal{R}^e$ at positions not in $P_{\mathcal{B}^e}(t)$.*

where $\mathcal{R}^e$ is the set of rules generated by all rules e' in $\vec{E}$ such that $e \succ_E e'$, and $\mathcal{B}^e = Right(\mathcal{R}^e)$. We denote by $\mathcal{R}$ the set of rules generated by all rules in $\vec{E}$, and we define $\mathcal{B} = Right(\mathcal{R})$ as the set of $\mathcal{B}$-terms.

We define the $\mathcal{B}$-positions *of t as all positions in $P_{\mathcal{B}}(t)$ and the* non-$\mathcal{B}$-positions *of t as all other positions of t.*

We denote by $\mathcal{B}^{\lhd s}$ the set of terms t in $\mathcal{B}$ s.t. $s \rhd t$.

Using a west-ordering (which includes the subterm relation) for comparing the left-hand sides of the rules is crucial in order to ensure that a rule in $\vec{E}$ is included in $\mathcal{R}$ if and only if it is irreducible at non-$\mathcal{B}$-positions by the other rules in $\mathcal{R}$.

The way we define $\mathcal{R}$ is quite similar to the definition in [BG94] and especially the one in [BGNR99]. The main difference is that only non-$\mathcal{B}$-positions are required to be irreducible. As a consequence $\mathcal{R}$ can be non-terminating and overlapping. However, if the ordering $\succ$ is a west-ordering, i.e. it fulfils the subterm property as well, taking $\succ_{\rhd}$ as $\succ$, the generated $\mathcal{R}$ coincides with the rewrite system generated in [BGNR99]. Note that in this case the right-hand sides of $\mathcal{R}$ are irreducible, which makes the fact that we only consider non-$\mathcal{B}$-positions irrelevant.

Example 4. For the closed set of equations in Example 3, $\mathcal{R}$ consists of the following two rules:

$$\begin{aligned} a &\to g(a) \\ g(g(a)) &\to g(a) \end{aligned}$$

Definition 9. *By $\vec{E}_{\mathcal{B}}$ we denote the subset of rules $l \to r$ in $\vec{E}$ s.t. $r \in \mathcal{B}$.*

Again following the example, $\vec{E}_{\mathcal{B}}$ includes the rules

$$\begin{aligned} a &\to g(a) \\ g^k(a) &\to g(a) \quad \text{for all } k \geq 2 \end{aligned}$$

The following lemmas state several relevant properties of $\mathcal{R}$, $\vec{E}_{\mathcal{B}}$ and $\mathcal{B}$, which will be used in the confluence proof, and in particular to show the so-called *return property*.

Lemma 4 roughly states that if a term is reducible (at non-$\mathcal{B}$-positions) at some point of the construction of $\mathcal{R}$ then it will stay irreducible from this point on. As a consequence all rules that have not been included in $\mathcal{R}$ are reducible at some non-$\mathcal{B}$-position by $\mathcal{R}$. In particular it happens for the left-hand sides of rules in $\vec{E}_{\mathcal{B}} \setminus \mathcal{R}$ (Lemma 5).

Lemma 4. *Let t be a term in $T(\mathcal{F})$ which is reducible by a rule $l \to r \in \mathcal{R}^e$ at positions not in $P_{\mathcal{B}^e}(t)$, for some rule e in $\vec{E}$. Then t is reducible by $l \to r$ at positions not in $P_{\mathcal{B}^d}(t)$ for all $d \succ_E e$ in $\vec{E}$.*

Lemma 5. *If $l \to r$ is a rule in $\vec{E}_{\mathcal{B}} \setminus \mathcal{R}$ then l is reducible by $\mathcal{R}$ at some non-$\mathcal{B}$-position.*

From these two previous lemmas we can infer the following property on $\mathcal{B}$-terms.

Lemma 6. *Let t be a term in $\mathcal{B}$. Then $t \notin Left(\vec{E}_{\mathcal{B}})$.*

Therefore it follows that there are no two different rules in $\vec{E}_{\mathcal{B}}$ with the same left-hand side (by showing contradiction with the previous lemma after applying a paramodulation step between the left-hand sides of both rules).

Finally all these previous properties let us prove the following property, which precisely[1] characterizes $\mathcal{R}$ in terms of $\vec{E}_{\mathcal{B}}$ and $\mathcal{B}$.

Lemma 7. *Let $l \to r$ be a rule in $\vec{E}_{\mathcal{B}}$. Then $l \to r \in \mathcal{R}$ if and only if l is irreducible by $\vec{E}_{\mathcal{B}}$ at every non-$\mathcal{B}$-position different from λ.*

This property generalizes the notion in [BGNR99] of irreducibility of rules of the ground TRS that defines the model. Here irreducibility amounts to irreducibility at non-$\mathcal{B}$-positions of the left and right-hand sides. Note that, in particular, if the right-hand sides of rules in $\mathcal{R}$ are irreducible by $\mathcal{R}$, then left-hand sides are irreducible as well, and both notions of irreducibility coincide.

4.1 The return property

In this section we prove the so-called *return property* which roughly states that if we reduce a $\mathcal{B}$-term t into a term u then there is a rewrite sequence from u to t again (without reducing any $\mathcal{B}$-subterm). This property is crucial to prove that $\mathcal{R}$ is confluent.

The intuition behind this property (at the ground level) comes from the fact that t is minimal wrt. $\succ$, and that moreover ordered paramodulation gives us $t_1 \simeq t_n$ for any chain $t_1 \overset{\succ}{\simeq} t_2 \cdots \overset{\succ}{\simeq} t_n$ of equations. Then, if t_n is minimal wrt. $\succ$ and it is reducible by some rule, for instance $t_1 \to t_n$ itself, at some position $p \neq \lambda$ (which may happen since $\succ$ can contradict the subterm property), we can obtain $t_1 \simeq t_n[t_n]_p$ by an ordered paramodulation step, which gives us that $t_n[t_n]_p$ is in the same chain of equations and, hence, $t_n[t_n]_p \overset{\succ}{\simeq} t_n$ is in the closure as well. In the example, since we have $a \overset{\succ}{\simeq} g(a)$ with $g(a)$ minimal and $g(a)$ reducible by $a \to g(a)$, we obtain $a \simeq g(g(a))$ and hence $g(g(a)) \overset{\succ}{\simeq} g(a)$. Thus if we reduce $g(a)$ with $a \to g(a)$, we can return to $g(a)$ using $g(g(a)) \to g(a)$.

Thanks to this return property, t_n can be chosen as a representative (not a normal form, since it can be reducible) of all $t_1, \ldots, t_n$. Moreover for every one

[1] Note that, by definition of $\mathcal{R}$ and $\vec{E}_{\mathcal{B}}$, we have $\mathcal{R} \subseteq \vec{E}_{\mathcal{B}}$.

of these terms t_i the representative can be obtained by an innermost rewrite sequence consisting in replacing every reducible subterm by its representative. These representatives are the $\mathcal{B}$-terms.

As usual, the proof becomes rather technical due to the reductions applied at positions that are below a variable in the equation in E which has generated the rule $l \to t_n$ in $\mathcal{R}$ (recall that the $\mathcal{B}$-terms are the right-hand sides of rules in $\mathcal{R}$). For this reason, many times we have to reduce in one shot all redexes involving the same subterm. This is denoted by a kind of parallel rewriting.

Now, we will give the main definitions and lemmas which allow us to prove the return property (all proofs can be found in [BR02]). First we define the set of reducible positions with respect to a given reduction relation.

Definition 10. *Let R be a TRS. Let $\curvearrowright$ denote a binary relation on abstracted terms wrt. R. And let $s \cdot \gamma$ be an abstracted term wrt. R. Then $red\_pos(s \cdot \gamma, \curvearrowright)$ denotes the set of positions $\bigcup_{t \cdot \delta} P_{\mathcal{X}}(t) \setminus P_{\mathcal{X}}(s)$ for all $t \cdot \delta$ such that $s \cdot \gamma \curvearrowright t \cdot \delta$.*

Definition 12 establishes a kind of innermost rewrite notion, which fairly mimics the aforementioned process for obtaining the representative of a given term. Since this special rewriting strategy is intended to be used in the proof of the return property, we are obliged to define it not only as an innermost, but also as a kind of parallel rewrite notion (Definition 11) in order to express substitution replacements at the non-ground level. Fortunately, due to the parallel rewriting strategy, every *illegal* subderivation (that is, a subderivation at or below a $\mathcal{B}$-position) can be omitted by applying the return property by induction on smaller $\mathcal{B}$-subterms (this is mainly what Lemma 9 states).

Definition 11. *Let R be a TRS with non-overlapped left-hand sides. Let $\curvearrowright$ denote a binary relation on abstracted terms wrt. R, such that $\curvearrowright \subseteq \curvearrowright_R$. We write $s \cdot \gamma \overset{\parallel}{\curvearrowright} t \cdot \delta$ if there is a derivation D of the form $s \cdot \gamma \overset{+}{\curvearrowright} t \cdot \delta$ and $t|_p \in \mathcal{X}$ for all $p \in red\_pos(s \cdot \gamma, \curvearrowright)$.*

Note that requiring $t|_p \in \mathcal{X}$ implies p to be a valid position in t and hence the rewrite steps not being overlapped.

By definition, every rewrite step with $\overset{\parallel}{\curvearrowright}$ represents a rewrite sequence with $\curvearrowright$.

Definition 12. *Let R be a TRS. Let $u \cdot \gamma$ be an abstracted term wrt. R. We write $u \cdot \gamma \rightsquigarrow_R v \cdot \delta$ if $u \cdot \gamma \overset{\parallel}{\curvearrowright}_{\underline{\{l \to r\}}} v \cdot \delta$ with $l \to r \in R$, $v \notin \mathcal{X}$, and there is no rule $s \to t \in R$ with $l \rhd s$ such that $u \cdot \gamma \curvearrowright_{\underline{\{s \to t\}}} w \cdot \rho$.*

In this case we say that $l \to r$ reduces *$\underline{u \cdot \gamma}$ wrt. $\rightsquigarrow_R$, and we refer to any derivation $u \cdot \gamma \overset{+}{\curvearrowright}_{\underline{\{l \to r\}}} v \cdot \delta$ which is equivalent to $u \cdot \gamma \overset{\parallel}{\curvearrowright}_{\underline{\{l \to r\}}} v \cdot \delta$ as a derivation with $\curvearrowright_R$* equivalent *to $s \cdot \gamma \rightsquigarrow_R v \cdot \delta$.*

Again, by definition, every non-empty rewrite sequence with $\rightsquigarrow_R$ (i.e. $u \cdot \gamma \overset{+}{\rightsquigarrow}_R w \cdot \rho$) represents a rewrite sequence with $\curvearrowright_R$ (i.e. $u \cdot \gamma \overset{+}{\curvearrowright}_R w \cdot \rho$). If D is a rewrite sequence which includes steps with $\curvearrowright_R$ and $\rightsquigarrow_R$ for some TRS R, we will write $D(\curvearrowright_R)$ to denote the corresponding rewrite sequence with only $\curvearrowright_R$ steps.

Lemma 8 is a rather simple but useful lemma, since it allows us to guarantee that during the transformation of a $\mathcal{B}$-term t towards itself in the proof of the return property, no tautology of the form $t \simeq t$ is obtained at the equational level till the last step. It is very important to show that we get back to t at the last step, since then all *illegal* steps on smaller $\mathcal{B}$-subterms are guaranteed to be already *repaired* and hence they can be omitted, as Lemma 9 states. Recall that $arg\_abs(t, R)$ is the result of abstracting as much as possible the arguments of t wrt. R (Definition 2), which implies that all proper $\mathcal{B}$-subterms of t are in the substitution part and, hence, blocked.

Lemma 8. *Let t be a term in $T(\mathcal{F})$ and p a position not in $P_{\mathcal{B}^{\triangleleft t}}(t)$. If $t \curvearrowright_{\vec{E}_{\mathcal{B}}} t[x]_p \cdot \delta \overset{*}{\curvearrowright}_{\vec{E}_{\mathcal{B}}} t' \cdot \gamma$ and $t' \notin \mathcal{X}$ then $t'\gamma \not\equiv t$.*

Lemma 9. *Let t be a term in $\mathcal{B}$. Let $\mathcal{S}$ be a subset of $\vec{E}_{\mathcal{B}}$ with non-overlapped left-hand sides. Let D be a rewrite sequence of the form $t \overset{!}{\curvearrowright}_{\mathcal{S}} t_1 \cdot \rho \overset{!}{\rightsquigarrow}_{\vec{E}_{\mathcal{B}}} t_m \cdot \beta$. Then there is a rewrite sequence D' of the form $arg\_abs(t, \mathcal{R}) \overset{*}{\curvearrowright}_{\vec{E}_{\mathcal{B}}} t'_m \cdot \beta'$ with $t'_m\beta' \equiv t_m\beta$ consisting of the steps in $D(\curvearrowright_{\vec{E}_{\mathcal{B}}})$ which are applied at positions not in $P_{\mathcal{B}^{\triangleleft t}}(t)$.*

The following lemma establishes the return property of $\mathcal{B}$-terms for rewrite sequences using $\vec{E}_{\mathcal{B}}$. Its proof, although based on the relatively simple intuition given at the beginning of this subsection, is quite long and technical, due mainly to the parallel rewriting issue combined with the need of explicity constructing a fair and complete return path to the initial term while avoiding tautologies.

We must point out that, although our aim is to prove the return property for rewrite sequences using $\mathcal{R}$ (Theorem 1), this weaker lemma is necessary since sometimes illegal steps can only be repaired by using rules in $\vec{E}_{\mathcal{B}} \setminus \mathcal{R}$.

Lemma 10. *Let t be a term in $\mathcal{B}$. Let $\mathcal{S}$ be a non-empty subset of $\vec{E}_{\mathcal{B}}$ with non-overlapped left-hand sides. If $t \overset{!}{\curvearrowright}_{\mathcal{S}} t_1 \cdot \rho \overset{!}{\rightsquigarrow}_{\vec{E}_{\mathcal{B}}} t_m \cdot \beta$ then $t_m\beta \equiv t$ or $t_m\beta \to t \in \vec{E}_{\mathcal{B}}$.*

Now, roughly we have to go from the return property with $\vec{E}_{\mathcal{B}}$ to the return property with $\mathcal{R}$. To this end we show that for some particular abstracted terms, the so-called *innermost reduced terms*, innermost rewrite sequences with $\curvearrowright_{\vec{E}_{\mathcal{B}}}$ are in fact rewrite sequences with $\curvearrowright_{\mathcal{R}}$ and preserve the innermost reduction property of the terms (Lemma 11). The proof of this lemma is based on Lemma 7.

Definition 13. *Let $s \cdot \gamma$ be an abstracted term wrt. $\mathcal{R}$. We say that $s \cdot \gamma$ is* innermost reduced *if there is no position $p \in red\_pos(s \cdot \gamma, \curvearrowright_{\vec{E}_{\mathcal{B}}})$ (i.e. redex in $s \cdot \gamma$) with $vars(s|_p) = \emptyset$ (i.e. which has never been reduced inside).*

Lemma 11. *Let $s \cdot \gamma$ be an abstracted term wrt. $\mathcal{R}$ which is innermost reduced and let D an innermost rewrite sequence of the form $s \cdot \gamma \overset{+}{\curvearrowright}_{\vec{E}_{\mathcal{B}}} t \cdot \delta$. Then D is a rewrite sequence with $\curvearrowright_{\mathcal{R}}$ and $t \cdot \delta$ is innermost reduced.*

Finally we obtain the return property of $\mathcal{B}$-terms for rewrite sequences using $\mathcal{R}$ by combining Lemmas 9 and 10: by carefully choosing the subset $\mathcal{S}$ of $\vec{E}_{\mathcal{B}}$ (and, in fact, of $\mathcal{R}$) for these lemmas, we obtain an abstracted term $t_1 \cdot \rho$ which is innermost reduced, and then we can show, by Lemma 11, that we have a return rewrite sequence using only $\mathcal{R}$. Finally, we can remove some steps of the resulting sequence showing that we have the sequence with $\mathcal{R}$ from $arg\_abs(t, \mathcal{R})$ (with the required initial step) to an abstraction of t again.

Theorem 1. *(return property) Let t be a term in $\mathcal{B}$. If $arg\_abs(t, \mathcal{R}) \curvearrowright_{\mathcal{R}} u \cdot \gamma$ then $u \cdot \gamma \overset{!}{\curvearrowright}_{\mathcal{R}} v \cdot \rho$ with $v\rho \equiv t$.*

4.2 Rewrite Proofs for the equational case

First we prove the confluence of the rewrite relation $\overset{*}{\curvearrowright}_{\mathcal{R}}$, which is essential for showing that we have $\mathcal{B}$-rewrite proofs for all consequences of the set of equations.

Using the return property we can show that if we reduce an abstraction of a left-hand side in $\mathcal{R}$ (not at top position) then there is a $\mathcal{B}$-rewrite sequence going back to (another abstraction of) the left-hand side again. The proof is based on the fact that left-hand sides can only be reducible by the other rules at $\mathcal{B}$-positions.

Lemma 12. *Let $s \cdot \gamma$ be an abstracted term such that $s\gamma \in Left(\mathcal{R})$. If there is a position $p \neq \lambda$, such that $s|_p\gamma \equiv l$ for some rule $l \to r$ in $\mathcal{R}$, then $s[x]_p \cdot (\gamma \cup \{x \mapsto r\} \overset{+}{\curvearrowright}_{\mathcal{R}} s' \cdot \gamma'$, where s' is not a variable and $s\gamma \equiv s'\gamma'$.*

From the previous lemma we know that we can close the only potential "picks" that we may have with $\curvearrowright_{\mathcal{R}}$. Therefore, local confluence follows.

Lemma 13. *(Local confluence) Let $s \cdot \gamma$ be an abstracted term. If $t \cdot \gamma_1 \curvearrowleft_{\mathcal{R}} s \cdot \gamma \curvearrowright_{\mathcal{R}} u \cdot \gamma_2$ then $t \cdot \gamma_1 \overset{*}{\curvearrowright}_{\mathcal{R}} v \cdot \delta \overset{*}{\curvearrowleft}_{\mathcal{R}} u \cdot \gamma_2$ for some abstracted term $v \cdot \delta$.*

Since $\curvearrowright_{\mathcal{R}}$ is terminating and locally confluent, it is confluent.

Theorem 2. *(Confluence) $\curvearrowright_{\mathcal{R}}$ is confluent.*

Now we prove that all equations in E have a $\mathcal{B}$-rewrite proof using $\mathcal{R}$, which also implies that $\mathcal{R}^*$ is a model for E.

First, we show that there is a $\mathcal{B}$-rewrite proof for all instances which are irreducible (at non-$\mathcal{B}$-positions) in the substitution part. A substitution σ is irreducible by $\mathcal{R}$ at non-$\mathcal{B}$-positions in a term t if, for all variables $x \in Dom(\sigma)$, $x\sigma$ is irreducible by $\mathcal{R}$ at non-$\mathcal{B}$-positions or x occurs in t only at $\mathcal{B}$-positions of $t\sigma$. A substitution σ is irreducible by $\mathcal{R}$ at non-$\mathcal{B}$-positions in an equation $s \simeq t$ if σ is irreducible by $\mathcal{R}$ at non-$\mathcal{B}$-positions in s and t. This irreducibility property is roughly inherited by the conclusion in paramodulation inferences with rules in $\mathcal{R}$.

Lemma 14. *Let $s \simeq t$ be an equation in E and σ a substitution irreducible at non-$\mathcal{B}$-positions in $s \simeq t$. If $u_1 \cdot \gamma_1$ is an abstraction of $s\sigma$ wrt. $\mathcal{R}$ and $u_2 \cdot \gamma_2$ is an abstraction of $t\sigma$ wrt. $\mathcal{R}$ then $u_1 \cdot \gamma_1 \overset{*}{\curvearrowright}_{\mathcal{R}} u \cdot \gamma \overset{*}{\curvearrowleft}_{\mathcal{R}} u_2 \cdot \gamma_2$ for some abstracted term $u \cdot \gamma$.*

Second, by first normalizing the substitution wrt. $\overset{*}{\curvearrowleft}_{\mathcal{R}}$, which gives us an irreducible instance, we have that there are $\mathcal{B}$-rewrite proofs for all instances.

Lemma 15. *Let $s \simeq t$ be an equation in $gnd(E)$. Then $s \overset{*}{\curvearrowright}_{\mathcal{R}} u \cdot \gamma \overset{*}{\curvearrowleft}_{\mathcal{R}} t$ for some abstracted term $u \cdot \gamma$.*

The following theorem follows directly from the previous lemma. Note that rewriting with $\curvearrowright_{\mathcal{R}}$ provides also a rewrite sequence with $\rightarrow_{\mathcal{R}}$.

Theorem 3. *$\mathcal{R}^* \models E$.*

Now, together with confluence of $\overset{*}{\curvearrowright}_{\mathcal{R}}$, we can show that all consequences of E have a $\mathcal{B}$-rewrite proof.

Lemma 16. *$E \models s \simeq t$ implies $s \overset{*}{\curvearrowright}_{\mathcal{R}} u \cdot \gamma \overset{*}{\curvearrowleft}_{\mathcal{R}} t$ for some abstracted term $u \cdot \gamma$.*

This result implies that we have produced a Knuth-Bendix completion procedure for $\mathcal{B}$-rewriting, i.e. for ordered rewriting with well-founded orderings.

5 The Horn case

In this section we generalise the results of the previous section to Horn clauses. In the following inference system it is assumed that in each clause with a non-empty antecedent one of the negative equations, the one that is written underlined, has been *selected* (see, e.g., [BG98]). In the Horn case this leads to positive unit strategies: left premises of paramodulations are unit clauses, and the only inferences involving non-unit clauses are equality resolution or paramodulation left on its selected equation.

Definition 14. *The inference system $\mathcal{H}$ for Horn clauses with respect to the well-founded ordering $\succ$ is defined as follows:*

paramodulation right:

$$\frac{\rightarrow l \simeq r \qquad \rightarrow s \simeq t}{\rightarrow (s[r]_p \simeq t)\sigma} \qquad \textit{where } \sigma = mgu(l, s|_p)$$

paramodulation left:

$$\frac{\rightarrow l \simeq r \qquad \Gamma, \underline{s \simeq t} \rightarrow \Delta}{(\Gamma, s[r]_p \simeq t \rightarrow \Delta)\sigma} \qquad \textit{where } \sigma = mgu(l, s|_p)$$

equality resolution:

$$\frac{\Gamma, \underline{s \simeq t} \rightarrow \Delta}{(\Gamma \rightarrow \Delta)\sigma} \qquad \textit{where } \sigma = mgu(s, t)$$

where moreover in both paramodulation rules l is maximal in its premise (that is, for some ground substitution θ, it holds that $l\sigma\theta \succ r\sigma\theta$), and $s|_p$ is not a variable except in the paramodulation right inference when $s|_p = x \in vars(t)$, $p = \lambda$ and l is also a variable (as it happens in the equational case).

Definition 15. *Let S be a set of Horn clauses. We denote by E_S the subset of all positive unit clauses in S, that is, $E_S = \{ s \simeq t \mid \rightarrow s \simeq t \in S \}$, and we denote by $\mathcal{R}_S$ the set of rules generated by E_S as in Definition 8.*

First, it is shown that if S does not contain the empty clause then $\mathcal{R}_S^*$ is a model for all irreducible (at non-$\mathcal{B}$-positions) instances of clauses in S.

A substitution σ is irreducible by $\mathcal{R}_S$ at non-$\mathcal{B}$-positions in a clause $\Gamma \rightarrow \Delta$ if σ is irreducible at non-$\mathcal{B}$-positions in all equation $s \simeq t$ occurring in Γ or Δ.

Lemma 17. *Let S be a set of Horn clauses closed under $\mathcal{H}$. Then $\mathcal{R}_S^* \models C\sigma$ for all clauses C in S and substitutions σ that are irreducible by $\mathcal{R}_S$ at non-$\mathcal{B}$-positions in C.*

Now, as in the equational case, by first normalizing the substitution, we can prove the main theorem.

Theorem 4. *(refutation completeness of $\mathcal{H}$ for Horn clauses)*
Let S be a set of Horn clauses closed under $\mathcal{H}$. Then $\square \in S$ if, and only if, S is unsatisfiable.

6 Conclusion

In this paper we have proved that well-foundedness of the term ordering is sufficient for the completeness of ordered paramodulation on Horn clauses with equality.

Our completeness proof is based on the use of $\mathcal{B}$-rewriting, a new restricted form of rewriting which forbids further rewriting in the right-hand sides introduced in the rewrite steps. Interestingly, a by-product of our results is a Knuth-Bendix completion procedure for $\mathcal{B}$-rewriting, and its completeness.

$\mathcal{B}$-rewriting is closely related to the use of *basic* strategies [BGLS95,NR95], which we believe can be compatible with our results (the only part of the proof that may have some problems is the so called return property). In fact, we plan to work out the basicness results using the notion of *closures* defined for clauses in [BGLS95], since it coincides, at the level of clauses, with our abstracted terms (which, for now, we only use in the proofs).

We also have to carefully explore to what extent redundancy notions are applicable in our framework for ordered paramodulation. Note that from our proof only tautology deletion is clearly possible. Subsumption seems to be possible but it may interfere with the proof of the return property. Other redundancy notions like demodulation may only be applicable in very restricted cases.

Finally we have to study how to extend our results to full first-order clauses. Here the problem comes from the fact that, within the model generation proof technique, the completeness for general clauses is usually based on the fact that, at most one positive equation is satisfied in the generated (rewrite) model. Since, in our case, the ordering can contradict the subterm relation, we do not know how to obtain such a minimal model.

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References

[BD94] Leo Bachmair and Nachum Dershowitz. Equational inference, canonical proofs, and proof orderings. *J. of the Association for Computing Machinery*, 41(2):236–276, February 1994.

[BDH86] Leo Bachmair, Nachum Dershowitz, and Jieh Hsiang. Orderings for equational proofs. In *First IEEE Symposium on Logic in Computer Science (LICS)*, pages 346–357, Cambridge, Massachusetts, USA, June 16–18, 1986. IEEE Computer Society Press.

[BG94] Leo Bachmair and Harald Ganzinger. Rewrite-based equational theorem proving with selection and simplification. *Journal of Logic and Computation*, 4(3):217–247, 1994.

[BG98] Leo Bachmair and Harald Ganzinger. Equational reasoning in saturation-based theorem proving. In W. Bibel and P. Schmitt, editors, *Automated Deduction: A Basis for Applications*. Kluwer, 1998.

[BGLS95] L. Bachmair, H. Ganzinger, Chr. Lynch, and W. Snyder. Basic paramodulation. *Information and Computation*, 121(2):172–192, 1995.

[BGNR99] Miquel Bofill, Guillem Godoy, Robert Nieuwenhuis, and Albert Rubio. Paramodulation with non-monotonic orderings. In *14th IEEE Symposium on Logic in Computer Science (LICS)*, pages 225–233, Trento, Italy, July 2–5, 1999.

[BR02] Miquel Bofill and Albert Rubio. Well-foundedness is sufficient for completeness of Ordered Paramodulation. Long version, 2002. Available at `www.lsi.upc.es/~albert/papers.html`.

[DJ90] Nachum Dershowitz and Jean-Pierre Jouannaud. Rewrite systems. In Jan van Leeuwen, editor, *Handbook of Theoretical Computer Science*, volume B: Formal Models and Semantics, chapter 6, pages 244–320. Elsevier Science Publishers B.V., Amsterdam, New York, Oxford, Tokyo, 1990.

[HR91] J. Hsiang and M Rusinowitch. Proving refutational completeness of theorem proving strategies: the transfinite semantic tree method. *Journal of the ACM*, 38(3):559–587, July 1991.

[NR95] Robert Nieuwenhuis and Albert Rubio. Theorem Proving with Ordering and Equality Constrained Clauses. *Journal of Symbolic Computation*, 19(4):321–351, April 1995.

[NR01] Robert Nieuwenhuis and Albert Rubio. Paramodulation-based theorem proving. In J.A. Robinson and A. Voronkov, editors, *Handbook of Automated Reasoning*. Elsevier Science Publishers and MIT Press, 2001.

[Wec91] W. Wechler. *Universal Algebra for Computer Scientists*, volume 25 of *EATCS Monographs on Theoretical Computer Science*. Springer-Verlag, Berlin, 1991.