On the Standard Linear Viscoelastic model

M. Pellicer (Universitat de Girona)

Work in collaboration with:

J. Solà-Morales (Universitat Politècnica de Catalunya) (bounded problem)
B. Said-Houari (Alhosn University, UAE) (problem in $\mathbb{R}^N$)

Trobada d'EDP's i Aplicacions

Universitat de Girona
10th June 2016

Outline

1. Introduction and modeling
   - Mechanical approach: SLV model
   - Acoustic approach: MGT equation
2. SLV in a bounded domain
   - Description of the spectrum of A
   - Main result 1: new scalar product and normality of A
   - Main result 2: optimal exponential decay
3. SLV in $\mathbb{R}^N$
   - The decay result
   - The eigenvalues expansion method

Outline

1. Introduction and modeling
   - Mechanical approach: SLV model
   - Acoustic approach: MGT equation
2. SLV in a bounded domain
   - Description of the spectrum of A
   - Main result 1: new scalar product and normality of A
   - Main result 2: optimal exponential decay
3. SLV in $\mathbb{R}^N$
   - The decay result
   - The eigenvalues expansion method

Mechanical approach: SLV model

Deformation of a **viscoelastic material**
(properties: viscous fluidity + elastic solidity)





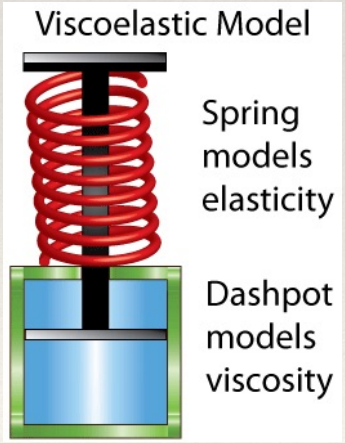
Mechanical approach: SLV model

Deformation of a **viscoelastic material**
(properties: viscous fluidity + elastic solidity)





Rheological approach (1D):



Rheological approach (1D):

- relation between σ (**stress**, force) and ε (**strain**, deformation)
- combination of

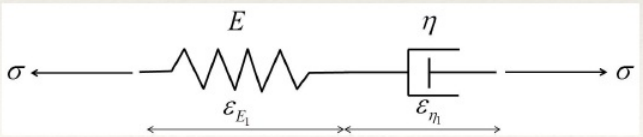
spring (elasticity) and **dashpot** (viscosity)

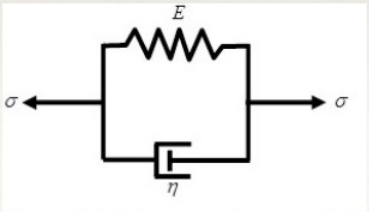
$$\sigma = E\,\varepsilon \qquad \sigma = \eta\,\varepsilon'$$

in **series** or in **parallel**

$$\varepsilon = \varepsilon_e + \varepsilon_v \qquad \varepsilon = \varepsilon_e = \varepsilon_v$$

$$\sigma = \sigma_e = \sigma_v \qquad \sigma = \sigma_e + \sigma_v$$

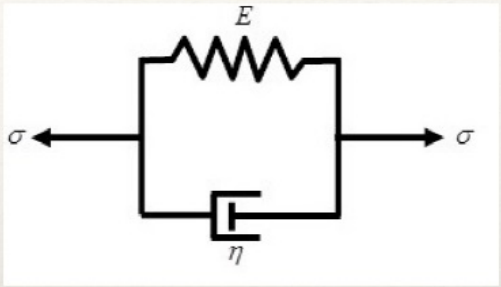




- mechanical tests: creeping and stress relaxation

Simplest models:

1. **Kelvin-Voigt**:
   - spring + dashpot in **parallel**
   - creeping, not stress relaxation
   - wave equation with **strong** damping
   - E.g. hard rubber



2. **Maxwell-model**:
   - spring + dashpot in **series**
   - stress relaxation, not creeping
   - wave equation with **weak** damping
   - E.g. gel

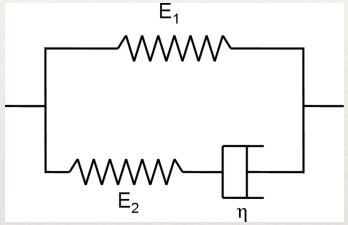


The Standard Linear Viscoelastic model

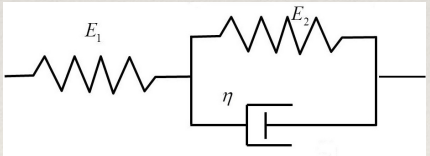
- a more realistic approach
- the most simple exhibiting creeping and stress relaxation

The Standard Linear Viscoelastic model

- a more realistic approach
- the most simple exhibiting creeping and stress relaxation
- 2 rheological approaches



spring + Maxwell in **parallel** (3-parameter, generalized Maxwell, ...)



spring + Kelvin-Voigt in **series** (SL solid, Zener, Kelvin, ...)

Constitutive equation:

$$\sigma + \alpha\sigma' = E\left(\varepsilon + \beta\varepsilon'\right)$$

(same equation with both approaches, but different relation of the coefficients with the physical parameters)

Constitutive equation:

$$\sigma + \alpha\sigma' = E\left(\varepsilon + \beta\varepsilon'\right)$$

(same equation with both approaches, but different relation of the coefficients with the physical parameters)

The continuous model (Standard Linear Viscoelastic model):

$$\alpha u_{ttt} + u_{tt} - a^2\left(u_{xx} + \beta u_{txx}\right) = 0,\ x \in [0, L],\ t \geq 0$$

where $u(x, t) =$ **deformation**, $\alpha, \beta > 0$

Acoustic approach: MGT equation

- Propagation of **sound**
- Acoustic velocity potential in viscous thermally relaxing fluids
- Medical and industrial application of **high intensity focused ultrasound** (lithotripsy, thermotherapy, ultrasound cleaning, . . .)

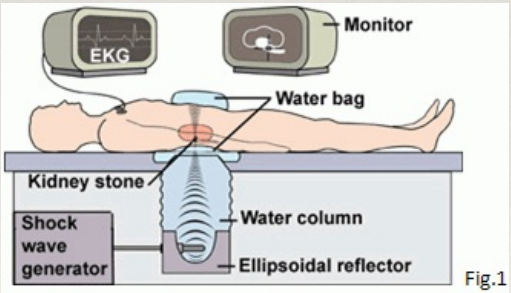
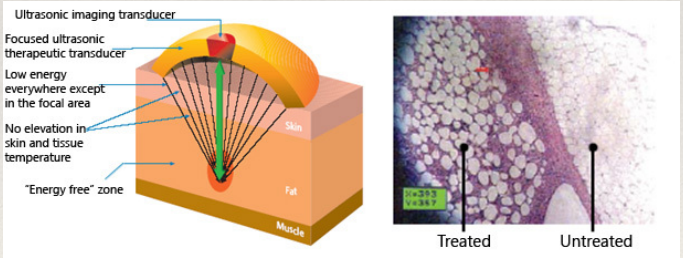


Fig.1



- Conservation of mass + momentum + energy
- Second order approximations
- Fourier law for heat conduction: $q = -K\nabla\theta$

$$\Downarrow$$

Kuznetsov equation (classic nonlinear Acoustics)

$$u_{tt} - c^2\Delta u - b\Delta u_t = \frac{\partial}{\partial t}\left(\frac{1}{c^2}\frac{B}{2A}(u_t)^2 + |\nabla u|^2\right)$$

(others: Westervelt equation, KZK equation, ...)

Problem: infinite speed of propagation !!!
(no good for previous applications)

Alternative $\Rightarrow$ **Maxwell-Cattaneo law**:

$$\alpha q_t + q = -K\nabla\theta$$

(α: relaxation time of heat flux $\Rightarrow$ thermal inertia therm)

$$\Downarrow$$

Jordan-Moore-Gibson-Thompson equation:

$$\alpha u_{ttt} + u_{tt} - c^2\Delta u - b\Delta u_t = \frac{\partial}{\partial t}\left(\frac{1}{c^2}\frac{B}{2A}(u_t)^2 + |\nabla u|^2\right)$$

Alternative $\Rightarrow$ **Maxwell-Cattaneo law**:

$$\alpha q_t + q = -K\nabla\theta$$

(α: relaxation time of heat flux $\Rightarrow$ thermal inertia therm)

$$\Downarrow$$

Jordan-Moore-Gibson-Thompson equation:

$$\alpha u_{ttt} + u_{tt} - c^2\Delta u - b\Delta u_t = \frac{\partial}{\partial t}\left(\frac{1}{c^2}\frac{B}{2A}(u_t)^2 + |\nabla u|^2\right)$$

Linear part $\Rightarrow$ **Moore-Gibson-Thompson equation**:

$$\boxed{\alpha u_{ttt} + u_{tt} - c^2\Delta u - b\Delta u_t = 0}$$

References

- (M) G. C. Gorain, (Proc. Indian Acad. Sci. (Math. Sci.), 2010)
- (M) M.S. Alves, C. Buriol, M.V. Ferreira, J.E. Muñoz Rivera, M. Sepúlveda, O. Vera (JMAA, 2013)
- (M) J.A. Conejero, C. Lizama, F. Ródenas (App. Math. Inf. Sciences, 2015)

- (A) B. Kaltenbacher, I. Lasiecka, R. Marchand (Control Cybernet., 2011)
- (A) B. Kaltenbacher, I. Lasiecka, M. K Pospieszalska (M^3ASc, 2012)
- (A) R. Marchand, T. McDevitt, R. Triggiani (M^3ASc, 2012)

Outline

1. Introduction and modeling
   - Mechanical approach: SLV model
   - Acoustic approach: MGT equation
2. SLV in a bounded domain
   - Description of the spectrum of A
   - Main result 1: new scalar product and normality of A
   - Main result 2: optimal exponential decay
3. SLV in $\mathbb{R}^N$
   - The decay result
   - The eigenvalues expansion method

The problem

Consider

$$\alpha u_{ttt} + u_{tt} - a^2 \Delta u - a^2 \beta \Delta u_t = 0, \ x \in \Omega \subset \mathbb{R}^N$$

with homogeneous Dirichlet b.c.

The problem

Consider

$$\alpha u_{ttt} + u_{tt} - a^2 \Delta u - a^2 \beta \Delta u_t = 0, \ x \in \Omega \subset \mathbb{R}^N$$

with homogeneous Dirichlet b.c.

In general:

$$\alpha\, u_{ttt} + u_{tt} + L\,(u + \beta u_t) = 0, \ x \in \Omega \subset \mathbb{R}^N$$

- Ω bounded, regular domain
- L self-adjoint, > 0, compact resolvent operator in H (Hilbert) *(eigenvalues:* $0 < \mu_1 \leq \mu_2 \leq \ldots \to \infty$*, semi-simple eigenfunctions:* ϕ_n *orthonormal)*
- several possible functional spaces $\mathcal{H}_i$ *(see [M-McD-T, 2012])*

Important remarks

$$\alpha\, u_{ttt} + u_{tt} - a^2 \Delta u - a^2 \beta \Delta u_t = 0,\ x \in \Omega \subset \mathbb{R}^N$$

- Although linear, this equation displays a variety of physical behaviors depending on the parameter values (**non-existence**, **existence but chaotic**, **exponential decay**)

Important remarks

$$\alpha\, u_{ttt} + u_{tt} - a^2 \Delta u - a^2 \beta \Delta u_t = 0,\ x \in \Omega \subset \mathbb{R}^N$$

- Although linear, this equation displays a variety of physical behaviors depending on the parameter values (**non-existence**, **existence but chaotic**, **exponential decay**)
- $\alpha > 0 \Rightarrow$ hyperbolic type equation

Important remarks

$$\alpha\, u_{ttt} + u_{tt} - a^2 \Delta u - a^2 \beta \Delta u_t = 0,\ x \in \Omega \subset \mathbb{R}^N$$

- Although linear, this equation displays a variety of physical behaviors depending on the parameter values (**non-existence**, **existence but chaotic**, **exponential decay**)
- $\alpha > 0 \Rightarrow$ hyperbolic type equation
- **Strong damping term (β)**:
  - $\alpha = 0$ (wave equation with strong damping): **regularizing effect** (hyperbolic to parabolic behavior)
  - $\alpha > 0$ (SLV equation): **WELL-POSEDNESS**

What is known (existence and decay)? (previous references)

$$\alpha\, u_{ttt} + u_{tt} - a^2 \Delta u - a^2 \beta \Delta u_t = 0,\ x \in \Omega \subset \mathbb{R}^N$$

What is known (existence and decay)? (previous references)

$$\alpha\, u_{ttt} + u_{tt} - a^2 \Delta u - a^2 \beta \Delta u_t = 0,\ x \in \Omega \subset \mathbb{R}^N$$

Well-posedness (β)

- $\beta = 0$ (and L unbounded): $\nexists$ C^0-semigroup
- $\beta > 0$: $\exists$ C^0-group.

What is known (existence and decay)? (previous references)

$$\alpha\, u_{ttt} + u_{tt} - a^2 \Delta u - a^2 \beta \Delta u_t = 0,\ x \in \Omega \subset \mathbb{R}^N$$

Well-posedness (β)

- $\beta = 0$ (and L unbounded): $\nexists$ C^0-semigroup
- $\beta > 0$: $\exists$ C^0-group.

Decay (α/β)

- $\alpha < \beta$ **(dissipative case)**: **exponentially stable group**
- $\alpha = \beta$ (conservative case): constant energy
- $\alpha > \beta$: chaotic behaviour

What is known (existence and decay)? (previous references)

$$\alpha\, u_{ttt} + u_{tt} - a^2 \Delta u - a^2 \beta \Delta u_t = 0,\ x \in \Omega \subset \mathbb{R}^N$$

Well-posedness (β)

- $\beta = 0$ (and L unbounded): $\nexists$ C^0-semigroup
- $\beta > 0$: $\exists$ C^0-group.

Decay (α/β)

- $\alpha < \beta$ **(dissipative case)**: **exponentially stable group**
- $\alpha = \beta$ (conservative case): constant energy
- $\alpha > \beta$: chaotic behaviour

Remark: third hidden parameter a^2 through the spectrum of $L = -a^2\Delta$ (eigenvalues: μ_n)

What we do?

- We focus in the **dissipative case** $\alpha < \beta$

What we do?

- We focus in the **dissipative case** $\alpha < \beta$
- Using a careful analysis of the spectrum of A we ...
  - Improve the **description of the spectrum** of [MMcDT, 2012]

What we do?

- We focus in the **dissipative case** $\alpha < \beta$
- Using a careful analysis of the spectrum of A we ...
  - Improve the **description of the spectrum** of [MMcDT, 2012]
  - Find a **new and explicit scalar product** that makes A **normal** (almost always)

What we do?

- We focus in the **dissipative case** $\alpha < \beta$
- Using a careful analysis of the spectrum of A we ...
  - Improve the **description of the spectrum** of [MMcDT, 2012]
  - Find a **new and explicit scalar product** that makes A **normal** (almost always)
  - Give the **optimal exponential decay rate** of solutions

  (previous work for wave equation with strong damping: M.P., J. Solà-Morales in JDE 2009)

Description of the spectrum of A

We write it as a first order system:

$$\frac{dU}{dt} = AU \ , \ U \in D(A), \ \text{with } AU = \begin{pmatrix} v \\ w \\ -\frac{1}{\alpha} L(u + \beta v) - \frac{1}{\alpha} w \end{pmatrix}$$

The eigenvalues:

Characteristic equation:

$$\alpha\lambda^3 + \lambda^2 + \beta\mu_n\lambda + \mu_n = 0$$

Cubic equation $\Rightarrow$ 3 solutions for each n:

$$\lambda_1^n, \lambda_2^n, \lambda_3^n$$

(type depending on the sign of Cardano's discriminant, whose zeroes are m_1, m_2)

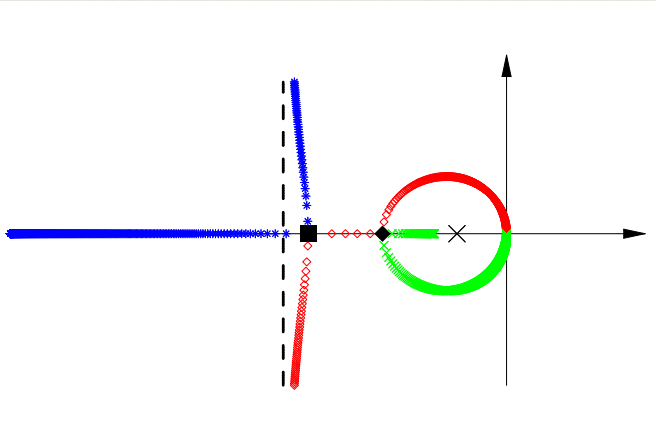
In general, **1 real** sequence λ_1^n and **2 nonreal** ones, $\lambda_2^n = \overline{\lambda_3^n}$

In general, **1 real** sequence λ_1^n and **2 nonreal** ones, $\lambda_2^n = \overline{\lambda_3^n}$

EXCEPT ...

if $\alpha/\beta < 1/9$ (**overdamping**) and $\mu_n \in [m_1, m_2]$ (finite number!):

- 3 real (if $\mu_n \in (m_1, m_2)$)
- 1 real, 2 real double (if $\mu_n = m_1$ or $\mu_n = m_2$)
  (actually triple root if $\alpha/\beta = 1/9$ and $\mu_n = m_1 = m_2$)



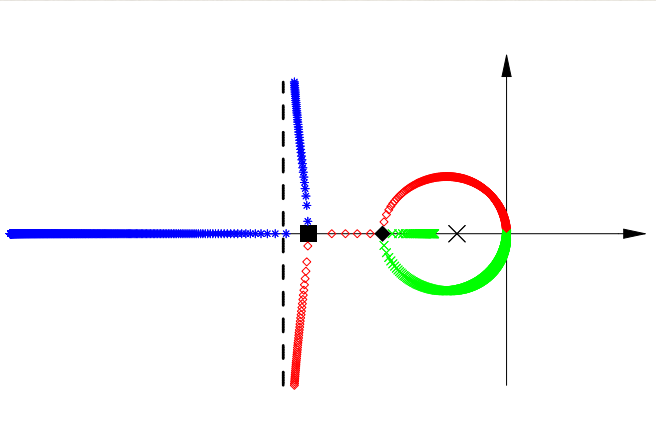
Spectrum when $\alpha/\beta < 1/9$

In general, **1 real** sequence λ_1^n and **2 nonreal** ones, $\lambda_2^n = \overline{\lambda_3^n}$

EXCEPT ...

if $\alpha/\beta < 1/9$ (**overdamping**) and $\mu_n \in [m_1, m_2]$ (finite number!):

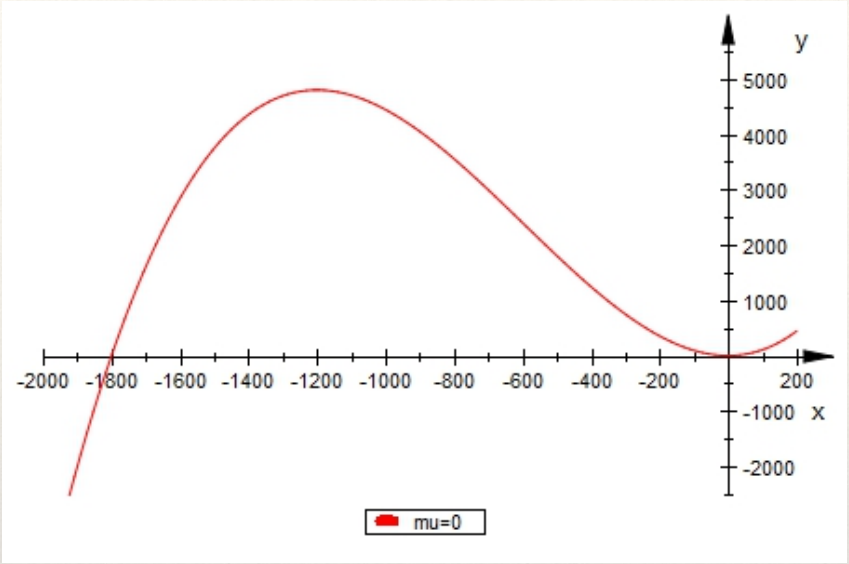
- 3 real (if $\mu_n \in (m_1, m_2)$)
- 1 real, 2 real double (if $\mu_n = m_1$ or $\mu_n = m_2$)
  (actually triple root if $\alpha/\beta = 1/9$ and $\mu_n = m_1 = m_2$)



Spectrum when $\alpha/\beta < 1/9$

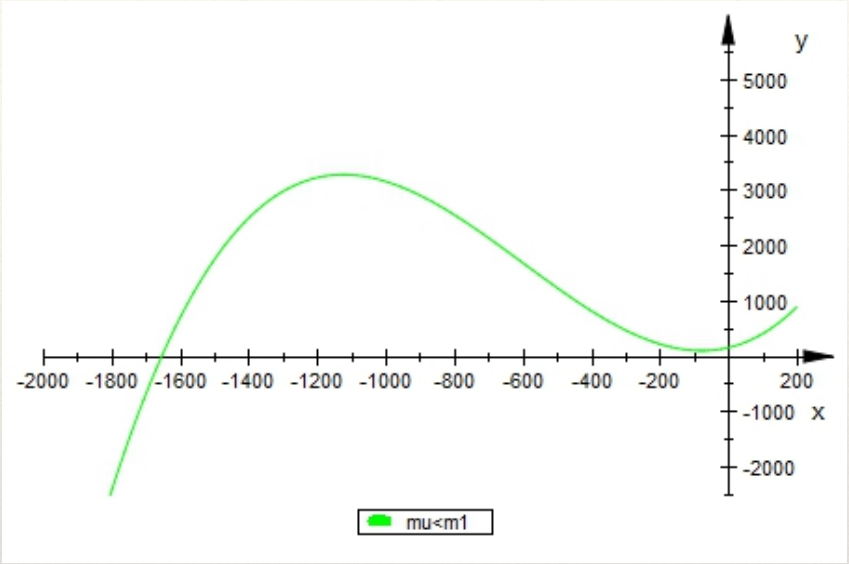
- The cases $m_1 \leq \mu_n \leq m_2$ are not considered in [MMcDT, 2012].
- In the **discrete case**, they can happen or not (depending on the parameters) (but they **DO** as $\alpha \to 0$!!)
- In the **continuous case (unbounded domain)** they **DO** happen.

Characteristic polynomial $\alpha\lambda^3 + \lambda^2 + \beta\mu_n\lambda + \mu_n = 0$ as n increases:



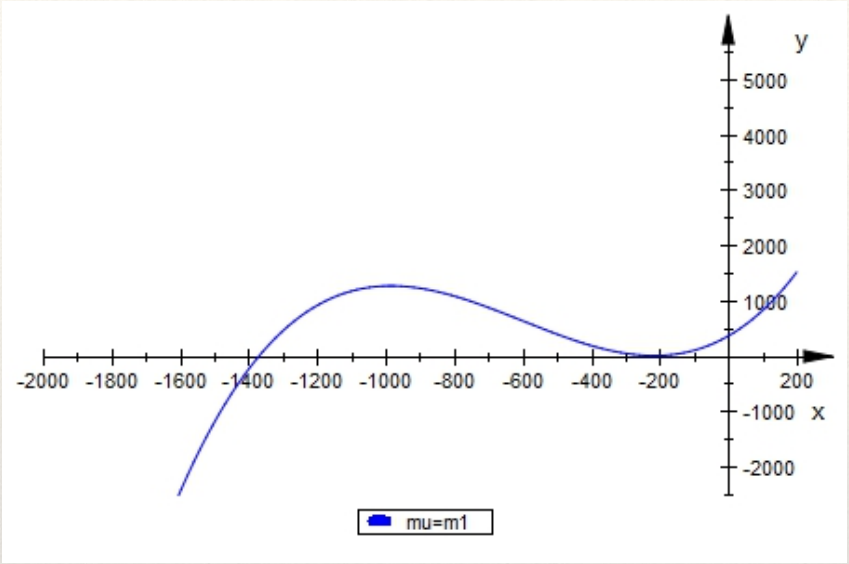
Case $0 < \frac{\alpha}{\beta} < \frac{1}{9}$

Characteristic polynomial $\alpha\lambda^3 + \lambda^2 + \beta\mu_n\lambda + \mu_n = 0$ as n increases:



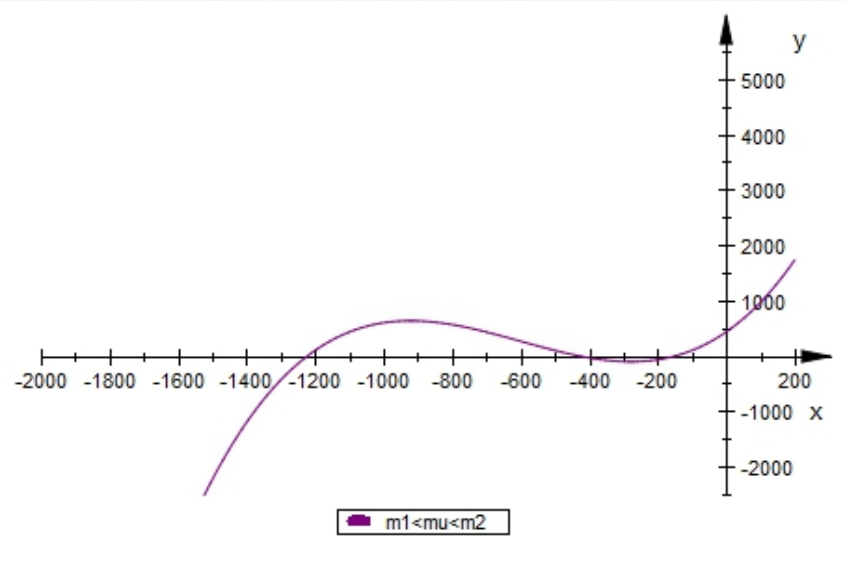
Case $0 < \frac{\alpha}{\beta} < \frac{1}{9}$

Characteristic polynomial $\alpha\lambda^3 + \lambda^2 + \beta\mu_n\lambda + \mu_n = 0$ as n increases:



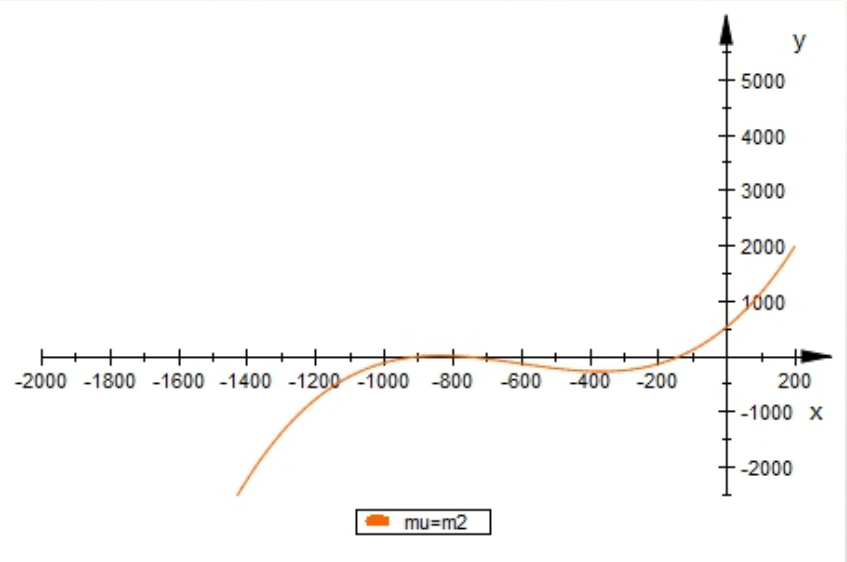
Case $0 < \frac{\alpha}{\beta} < \frac{1}{9}$

Characteristic polynomial $\alpha\lambda^3 + \lambda^2 + \beta\mu_n\lambda + \mu_n = 0$ as n increases:



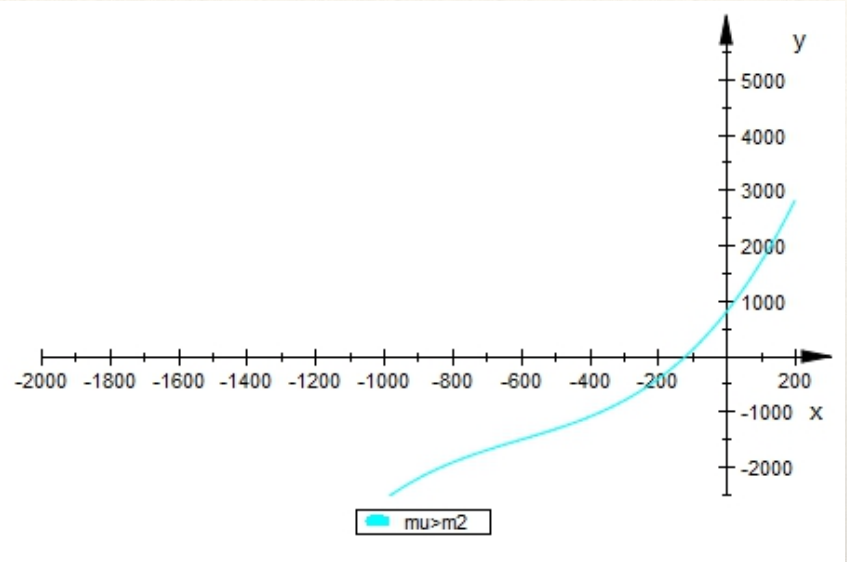
Case $0 < \dfrac{\alpha}{\beta} < \dfrac{1}{9}$

Characteristic polynomial $\alpha\lambda^3 + \lambda^2 + \beta\mu_n\lambda + \mu_n = 0$ as n increases:



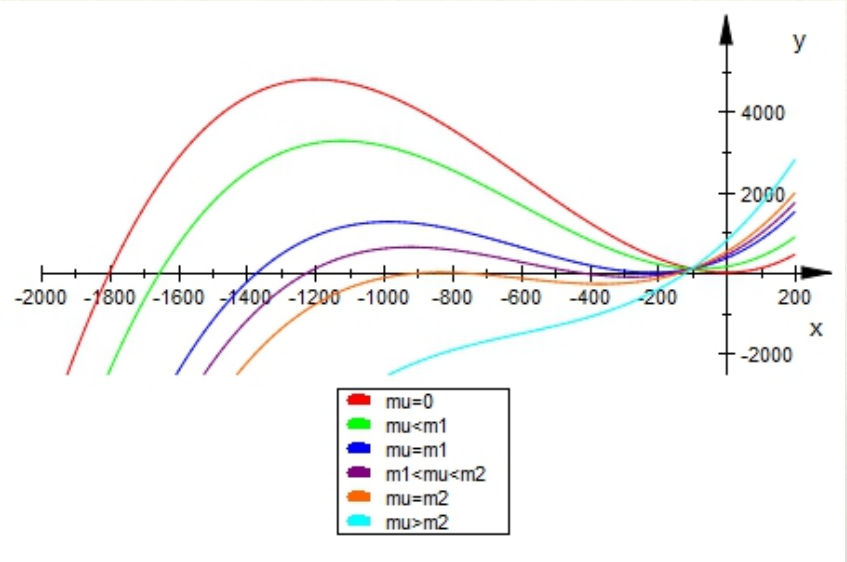
Case $0 < \frac{\alpha}{\beta} < \frac{1}{9}$

Characteristic polynomial $\alpha\lambda^3 + \lambda^2 + \beta\mu_n\lambda + \mu_n = 0$ as n increases:



Case $0 < \frac{\alpha}{\beta} < \frac{1}{9}$

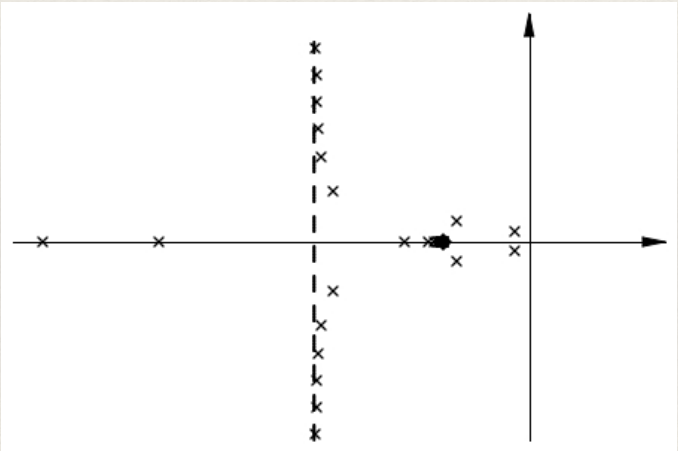
Characteristic polynomial $\alpha\lambda^3 + \lambda^2 + \beta\mu_n\lambda + \mu_n = 0$ as n increases:

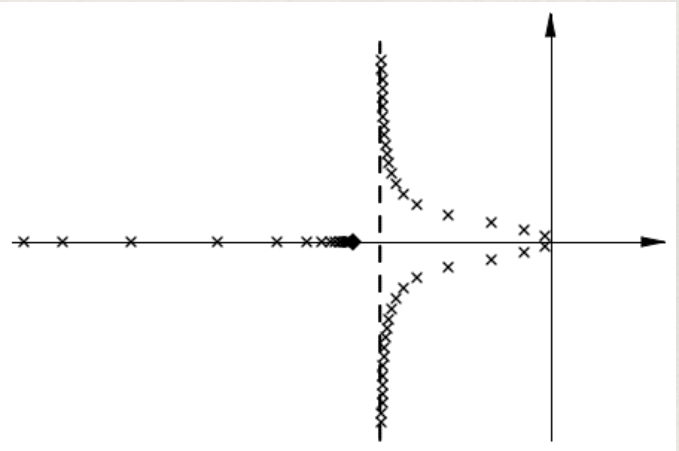


Case $0 < \frac{\alpha}{\beta} < \frac{1}{9}$

They fulfill:

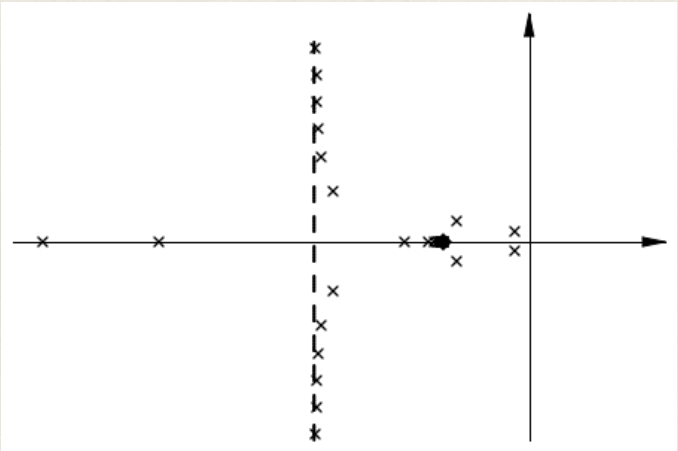
- $\lambda_k^n \in \mathbb{R} \Rightarrow \lambda_k^n \nearrow -\frac{1}{\beta}$ **(essential spectrum)**

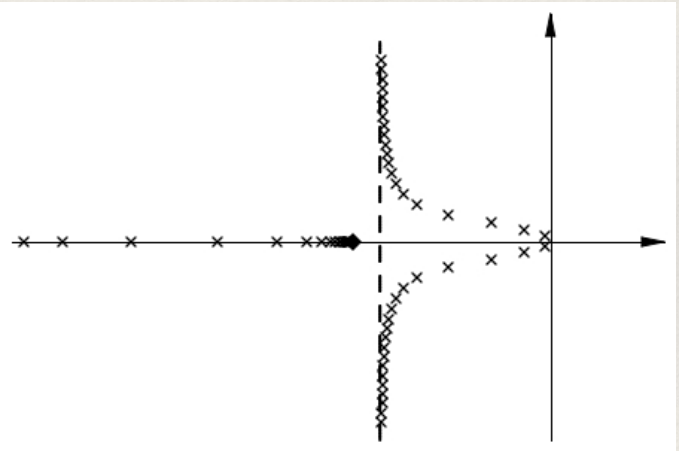




They fulfill:

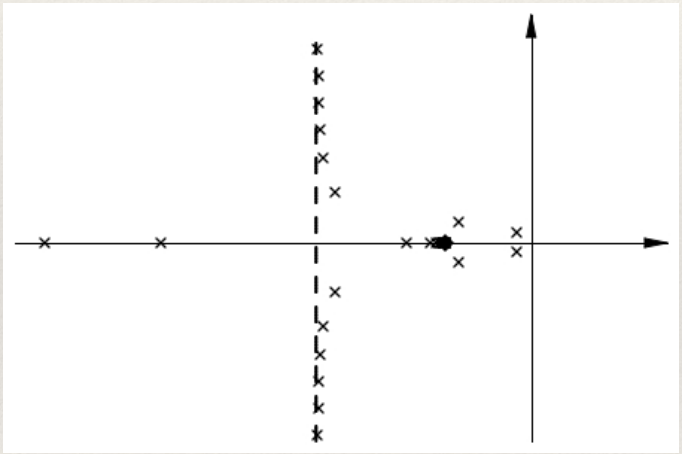
- $\lambda_k^n \in \mathbb{R} \Rightarrow \lambda_k^n \nearrow -\frac{1}{\beta}$ **(essential spectrum)**
- $\lambda_k^n \in \mathbb{C} \setminus \mathbb{R} \Rightarrow 0 > \mathrm{Re}(\lambda_k^n) \searrow -\frac{1}{2}\left(\frac{1}{\alpha} - \frac{1}{\beta}\right)$



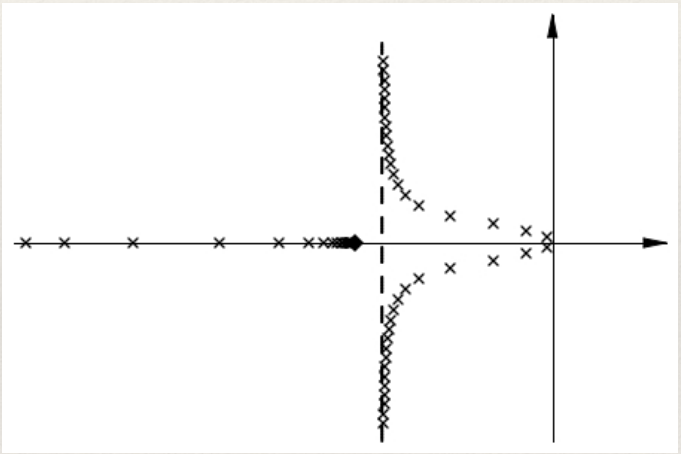


They fulfill:

- $\lambda_k^n \in \mathbb{R} \Rightarrow \lambda_k^n \nearrow -\frac{1}{\beta}$ **(essential spectrum)**
- $\lambda_k^n \in \mathbb{C} \setminus \mathbb{R} \Rightarrow 0 > \mathrm{Re}(\lambda_k^n) \searrow -\frac{1}{2}\left(\frac{1}{\alpha} - \frac{1}{\beta}\right)$
- $-\frac{1}{2}\left(\frac{1}{\alpha} - \frac{1}{\beta}\right) < -\frac{1}{\beta}$ if $\frac{\alpha}{\beta} < \frac{1}{3}$ (resp. $=$, $>$)



$\frac{\alpha}{\beta} < \frac{1}{3}$

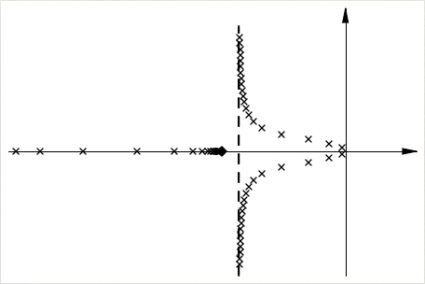


$\frac{\alpha}{\beta} > \frac{1}{3}$

Dominant spectrum

Two possibilities

- $-\frac{1}{\beta}$ (essential spectrum): no oscillations in the dominant part (**overdamped case**)
- $\{\lambda_2^1, \lambda_3^1\}$ (the first two nonreal eigenvalues)

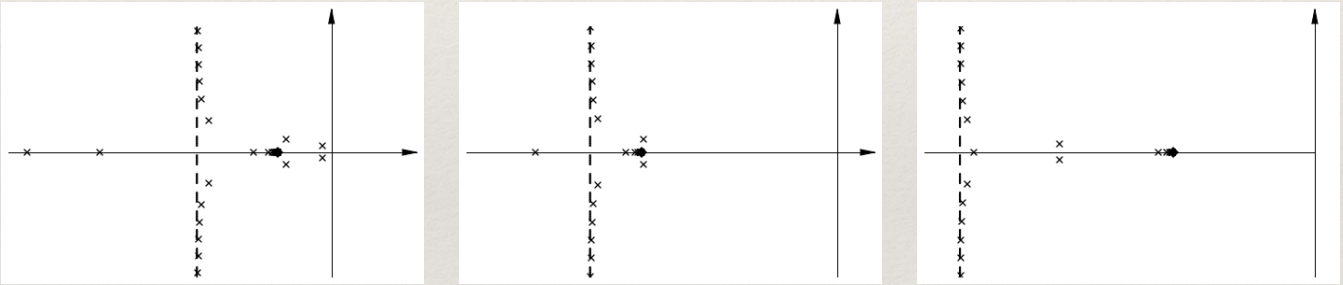


$$\frac{\alpha}{\beta} > \frac{1}{3} \Rightarrow \sigma_{max} = \mathrm{Re}(\lambda_2^1)$$

Dominant spectrum

Two possibilities

- $-\frac{1}{\beta}$ (essential spectrum): no oscillations in the dominant part (**overdamped case**)
- $\{\lambda_2^1, \lambda_3^1\}$ (the first two nonreal eigenvalues)



$$\frac{\alpha}{\beta} < \frac{1}{3} \Rightarrow \text{different possibilities}$$

(still not explicit condition: solve the characteristic equation for $\mu_n = \mu_1$)

Main result 1: new scalar product and normality of A

Theorem 1

- If $\mu_n \neq m_1,\ m_2$:
  - we construct a **new and explicit scalar product** $\langle \cdot, \cdot \rangle_G$ where A is a **normal** operator
  - the eigenfunctions are **orthonormal and complete** with G

Main result 1: new scalar product and normality of A

Theorem 1

- If $\mu_n \neq m_1,\ m_2$:
  - we construct a **new and explicit scalar product** $\langle \cdot, \cdot \rangle_G$ where A is a **normal** operator
  - the eigenfunctions are **orthonormal and complete** with G
- If any $\mu_n = m_1,\ m_2$, A **not normal** in any scalar product *(multiplicity of eigenvalues)*.

Main result 1: new scalar product and normality of A

Theorem 1

- If $\mu_n \neq m_1,\ m_2$:
  - we construct a **new and explicit scalar product** $\langle \cdot, \cdot \rangle_G$ where A is a **normal** operator
  - the eigenfunctions are **orthonormal and complete** with G
- If any $\mu_n = m_1,\ m_2$, A **not normal** in any scalar product *(multiplicity of eigenvalues)*.

Remarks

- A normal if $AA^* = A^*A$
  ($\Rightarrow$ diagonable, orthonormal eigenfunctions)

Main result 1: new scalar product and normality of A

Theorem 1

- If $\mu_n \neq m_1,\ m_2$:
  - we construct a **new and explicit scalar product** $\langle \cdot, \cdot \rangle_G$ where A is a **normal** operator
  - the eigenfunctions are **orthonormal and complete** with G
- If any $\mu_n = m_1,\ m_2$, A **not normal** in any scalar product *(multiplicity of eigenvalues)*.

Remarks

- A normal if $AA^* = A^*A$
  ($\Rightarrow$ diagonable, orthonormal eigenfunctions)
- A normal is the best we can expect
  (A is not self-adjoint as it has non-real eigenvalues)

Main result 1: new scalar product and normality of A

Theorem 1

- If $\mu_n \neq m_1,\ m_2$:
  - we construct a **new and explicit scalar product** $\langle \cdot, \cdot \rangle_G$ where A is a **normal** operator
  - the eigenfunctions are **orthonormal and complete** with G
- If any $\mu_n = m_1,\ m_2$, A **not normal** in any scalar product *(multiplicity of eigenvalues)*.

Remarks

- A normal if $AA^* = A^*A$
  ($\Rightarrow$ diagonable, orthonormal eigenfunctions)
- A normal is the best we can expect
  (A is not self-adjoint as it has non-real eigenvalues)
- Main difficulty: proving the equivalence of the new scalar product and the natural one

The idea ...

If A normal in $\langle , \rangle_G \Rightarrow$ A diagonable $\Rightarrow$ orthonormal eigenfunctions in this norm.

So, it is natural to define

$$G = \begin{pmatrix} G_1 & 0 & \cdots & 0 & \cdots \\ 0 & G_2 & \cdots & 0 & \cdots \\ \vdots & \vdots & \ddots & \vdots & \ddots \\ 0 & 0 & \cdots & G_n & \cdots \\ \vdots & \vdots & \ddots & \vdots & \ddots \end{pmatrix}$$

where $G_n = (\overline{C_n^{-1}})^T C_n^{-1}$, with $C_n = \mathrm{col}(\Psi_1^n, \Psi_2^n, \Psi_3^n)$ (eigenfunctions)

The idea ...

If A normal in $\langle , \rangle_G \Rightarrow$ A diagonable $\Rightarrow$ orthonormal eigenfunctions in this norm.

So, it is natural to define

$$G = \begin{pmatrix} G_1 & 0 & \cdots & 0 & \cdots \\ 0 & G_2 & \cdots & 0 & \cdots \\ \vdots & \vdots & \ddots & \vdots & \ddots \\ 0 & 0 & \cdots & G_n & \cdots \\ \vdots & \vdots & \ddots & \vdots & \ddots \end{pmatrix}$$

where $G_n = (\overline{C_n^{-1}})^T C_n^{-1}$, with $C_n = \mathrm{col}(\Psi_1^n, \Psi_2^n, \Psi_3^n)$ (eigenfunctions)

Remark 1

Observe that $\langle u, v \rangle_{G_n} = \ldots = \langle C_n^{-1} u, C_n^{-1} v \rangle_{\mathbb{R}^3}$

The idea ...

If A normal in $\langle , \rangle_G \Rightarrow$ A diagonable $\Rightarrow$ orthonormal eigenfunctions in this norm.

So, it is natural to define

$$G = \begin{pmatrix} G_1 & 0 & \cdots & 0 & \dots \\ 0 & G_2 & \cdots & 0 & \dots \\ \vdots & \vdots & \ddots & \vdots & \ddots \\ 0 & 0 & \cdots & G_n & \dots \\ \vdots & \vdots & \ddots & \vdots & \ddots \end{pmatrix}$$

where $G_n = (\overline{C_n^{-1}})^T C_n^{-1}$, with $C_n = \mathrm{col}(\Psi_1^n, \Psi_2^n, \Psi_3^n)$ (eigenfunctions)

Remark 2

$\langle , \rangle_{G_n} \sim \langle , \rangle$ **uniformly in** $n \Rightarrow$ G equivalent to the natural product

The idea ...

If A normal in $\langle , \rangle_G \Rightarrow$ A diagonable $\Rightarrow$ orthonormal eigenfunctions in this norm.

So, it is natural to define

$$G = \begin{pmatrix} G_1 & 0 & \cdots & 0 & \dots \\ 0 & G_2 & \cdots & 0 & \dots \\ \vdots & \vdots & \ddots & \vdots & \ddots \\ 0 & 0 & \cdots & G_n & \dots \\ \vdots & \vdots & \ddots & \vdots & \ddots \end{pmatrix}$$

where $G_n = (\overline{C_n^{-1}})^T C_n^{-1}$, with $C_n = \mathrm{col}(\Psi_1^n, \Psi_2^n, \Psi_3^n)$ (eigenfunctions)

Remark 3

Only possible when non (algebraically) multiple eigenvalues !!!
(in which case A **not normal** because not diagonable)

Example in $\mathcal{H} = H_0^1(\Omega) \times H_0^1(\Omega) \times L^2(\Omega)$*:*

$$O_n = \begin{pmatrix} \mu_n & 0 & 0 \\ 0 & \mu_n & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$G_n = \begin{pmatrix} \frac{2\alpha\beta^2+3\alpha+\beta}{2\alpha\beta^2}\mu_n + o(\mu_n) & \frac{\alpha+\beta}{2\alpha\beta}\mu_n + o(\mu_n) & \frac{4\alpha^2\beta^2+\beta^2+3\alpha^2}{4\alpha\beta^3}\cdot 1 + o(1) \\ \frac{\alpha+\beta}{2\alpha\beta}\mu_n + o(\mu_n) & \frac{\alpha+\beta}{2\alpha}\mu_n + o(\mu_n) & \frac{(\alpha+\beta)^2}{4\alpha\beta^2}\cdot 1 + o(1) \\ \frac{4\alpha^2\beta^2+\beta^2+3\alpha^2}{4\alpha\beta^3}\cdot 1 + o(1) & \frac{(\alpha+\beta)^2}{4\alpha\beta^2}\cdot 1 + o(1) & \frac{\alpha+\beta}{2\beta}\cdot 1 + o(1) \end{pmatrix}$$

Main result 2: optimal exponential decay

Theorem 2

- If A normal in $\langle \cdot, \cdot \rangle_G$:

$$\|U(t)\|_G \leq e^{\sigma_{max} t} \|U(0)\|_G$$

$\sigma_{max} = \mathrm{Re}(\lambda_2^1), -\frac{1}{\beta} < 0$, **dominant spectrum**, **optimal**.

Main result 2: optimal exponential decay

Theorem 2

- If A normal in $\langle \cdot, \cdot \rangle_G$:

$$\|U(t)\|_G \leq e^{\sigma_{max} t} \|U(0)\|_G$$

$\sigma_{max} = \mathrm{Re}(\lambda_2^1), -\frac{1}{\beta} < 0$, **dominant spectrum**, **optimal**.

- If A not normal, **ALSO true** (but in another norm)
(multiple eigenvalues, but not dominant)

Main result 2: optimal exponential decay

Theorem 2

- If A normal in $\langle \cdot, \cdot \rangle_G$:

$$\|U(t)\|_G \leq e^{\sigma_{max} t} \|U(0)\|_G$$

$\sigma_{max} = \mathrm{Re}(\lambda_2^1), -\frac{1}{\beta} < 0$, **dominant spectrum**, **optimal**.

- If A not normal, **ALSO true** (but in another norm) *(multiple eigenvalues, but not dominant)*

Remarks

In [MMcDT 2012]: exponential decay with the growth bound

$$\inf\{\omega \in \mathbb{R}; \|e^{At}\| \leq M_\omega e^{\omega t} \ \forall t \geq 0\} \quad \textit{(usual norm)}$$

We improve it: $M_\omega = 1$, and the infimum is a **minimum** (also when σ_{max} is the essential spectrum)

Idea...

If A normal in $\langle , \rangle_G$ ...

- Eigenfunctions are orthonormal in G and complete

Idea...

If A normal in $\langle , \rangle_G$ ...

- Eigenfunctions are orthonormal in G and complete

$$\Downarrow$$

In this basis:

$$\|U(t)\|_G^2 = \sum_{n,j} |d_j^n|^2 e^{2Re(\lambda_j^n)t} \leq \sum_{n,j} |d_j^n|^2 e^{2\sigma_{max} t} \ (t > 0). \ \checkmark$$

Idea...

If A normal in $\langle , \rangle_G$ ...

- Eigenfunctions are orthonormal in G and complete

$$\Downarrow$$

In this basis:

$$\|U(t)\|_G^2 = \sum_{n,j} |d_j^n|^2 e^{2Re(\lambda_j^n)t} \leq \sum_{n,j} |d_j^n|^2 e^{2\sigma_{max} t} \ (t > 0). \ \checkmark$$

- Optimality:
  - Clear if $\sigma_{max} = \mathrm{Re}(\lambda_2^1)$ ✓
  - Also if $\sigma_{max} = -\frac{1}{\beta}$, as we have $\lambda_1^n \to -\frac{1}{\beta}$ ✓

If A not normal ...

- Multiple eigenvalues are not a problem because **they are not dominant** $\Rightarrow$ we can essentially do the same as before

If A not normal ...

- Multiple eigenvalues are not a problem because **they are not dominant** $\Rightarrow$ we can essentially do the same as before
- $A = \begin{pmatrix} A_0 & 0 \\ 0 & A_1 \end{pmatrix}$ (A_1 the part of the multiple eigenvalues)

  For A_0 the same as before applies:

$$\|e^{A_0 t}\|_{G_0} \leq e^{\sigma_{max}(A_0)\, t} \leq e^{\sigma_{max}(A)\, t}$$

  For A_1 we can define $\langle \cdot, \cdot \rangle_{G_{1,\varepsilon}}$ such that (for a certain $\varepsilon > 0$)

$$\|e^{A_1 t}\|_{G_{1,\varepsilon}} \leq e^{(\sigma_{max}(A_1)+\varepsilon)\, t} \leq e^{-\frac{1}{\beta}\, t} \leq e^{\sigma_{max}(A)\, t}$$

  $\Rightarrow$ G' defined as the orthogonal extension of G_0, G_1 ✓

Outline

1. Introduction and modeling
   - Mechanical approach: SLV model
   - Acoustic approach: MGT equation
2. SLV in a bounded domain
   - Description of the spectrum of A
   - Main result 1: new scalar product and normality of A
   - Main result 2: optimal exponential decay
3. SLV in $\mathbb{R}^N$
   - The decay result
   - The eigenvalues expansion method

The problem

We consider

$$\alpha u_{ttt} + u_{tt} - \Delta u - \beta \Delta u_t = 0 \text{ in } \mathbb{R}^N$$

with $\alpha < \beta$ ($a = 1$ without loss of generality)

What we do?

- Well-posed ✓ (not done before)
- decay rate (using **eigenvalues expansion method**)

The decay result

Theorem:

For $N + j \geq 3$,

$$\begin{aligned} \left\|\partial_x^j u(t)\right\|_{L^2} \quad \leq \quad & C(\|u_0\|_{L^1} + \|u_1\|_{L^1} + \|u_2\|_{L^1})(1+t)^{1/2-N/4-j/2} \\ & + C(\|\partial_x^j u_0\|_{L^2} + \|\partial_x^j u_1\|_{L^2} + \|\partial_x^j u_2\|_{L^2})e^{-ct} \end{aligned}$$

with $c = \min\left\{\frac{1}{\beta}, |\mathrm{Re}(\lambda_{2,3}(\xi_{\nu_1}))|\right\}$

Decay: for all $N \geq 3$ and all $j \geq 0$
(similar formula for all j. In particular, $N = 1, 2$ decay only if $j > 3/2,\ 1/2$)

Regularity: same as the initial conditions (no regularity-loss type)

Idea: eigenvalues expansion method

Behaviour of the *eigenvalues* in the Fourier space

Observe that same description of the spectrum as in the bounded domain, but $|\xi|$ ($\sim \sqrt{\mu_n}$) being a continuous parameter

Idea: eigenvalues expansion method

Behaviour of the *eigenvalues* in the Fourier space

Observe that same description of the spectrum as in the bounded domain, but $|\xi|$ ($\sim \sqrt{\mu_n}$) being a continuous parameter

Why? Plancherel Theorem:

$$\left\|\partial_x^j u(t)\right\|_{L^2}^2 = \int_{\mathbb{R}^N} |\xi|^{2j} |\hat{u}(\xi, t)|^2 d\xi$$

Roughly speaking ... ODE in the Fourier space for each ξ

$$\hat{u}(\xi, t) = C_1(\xi) e^{\lambda_1(\xi) t} + e^{\mathrm{Re}(\lambda_2(\xi)) t} \left[C_2(\xi) \cos(\mathrm{Im}(\lambda_2(\xi)) t) + C_3(\xi) \sin(\mathrm{Im}(\lambda_2(\xi)) t)\right]$$

($C_i(\xi)$ using initial conditions)

Split the previous integral in Υ_L, Υ_M, Υ_H:

Split the previous integral in Υ_L, Υ_M, Υ_H:

- $\Upsilon_L = \{|\xi| < \nu_1 << 1\} \Rightarrow$ **decay** (as $\hat{u} \sim e^{\mathrm{Re}(\lambda(\xi))t}$)

$$\begin{cases} \lambda_1(\xi) \sim -\dfrac{1}{\alpha}, \\ \lambda_2(\xi) \sim i|\xi| - \dfrac{1}{2}(\beta-\alpha)|\xi|^2, \\ \lambda_3(\xi) \sim -i|\xi| - \dfrac{1}{2}(\beta-\alpha)|\xi|^2 \end{cases}$$

- $\Upsilon_M = \{\nu_1 \leq |\xi| \leq \nu_2\}$:

$$\mathrm{Re}(\lambda^n_{1,2,3}(\xi)) < -\min\left\{\frac{1}{\beta}, |\mathrm{Re}(\lambda_{2,3}(\xi_{\nu_1}))|\right\}$$

- $\Upsilon_H = \{|\xi| > \nu_2 >> 1\} \Rightarrow$ **regularity** (as $\hat{u} \sim |\xi|^m e^{-c|\xi|^{-\hat{c}}t}$)

$$\begin{cases} \lambda_1(\xi) \sim -\dfrac{1}{\beta}, \\ \lambda_2(\xi) \sim -\dfrac{\beta-\alpha}{2\beta\alpha} + i|\xi|\sqrt{\dfrac{\beta}{\alpha}}, \\ \lambda_3(\xi) \sim -\dfrac{\beta-\alpha}{2\beta\alpha} - i|\xi|\sqrt{\dfrac{\beta}{\alpha}} \end{cases}$$

Imposing initial conditions (and taking the dominant terms) ...

$|\xi|$ **small:**

$$
\begin{aligned}
|\hat{u}(\xi,t)| \quad \leq \quad & C_L \left(|\xi|^2 |\hat{u}_0| + |\xi|^2 |\hat{u}_1| + |\hat{u}_2| \right) e^{-\frac{1}{\alpha} t} \\
& + C_L \left(|\hat{u}_0| + |\xi|^2 |\hat{u}_1| + |\hat{u}_2| \right) e^{-\frac{\beta-\alpha}{2} |\xi|^2 t} \cos(|\xi| t) \\
& + C_L \left(|\xi| |\hat{u}_0| + \frac{1}{|\xi|} |\hat{u}_1| + \frac{1}{|\xi|} |\hat{u}_2| \right) e^{-\frac{\beta-\alpha}{2} |\xi|^2 t} \sin(|\xi| t)
\end{aligned}
$$

$$\Downarrow$$

(some integral bounds for $N + j \geq 3$ or for all $N \geq 1$)

$$\Downarrow$$

$$
\left(\int_{\Upsilon_L} |\xi|^{2j} |\hat{u}(\xi,t)|^2 d\xi \right)^{1/2} \leq C(\|u_0\|_{L^1}^2 + \|u_1\|_{L^1}^2 + \|u_2\|_{L^1}^2)(1+t)^{\frac{1}{2} - \frac{N}{4} - \frac{j}{2}}
$$

$|\xi|$ **medium** (and $|\xi| \neq \sqrt{m_1}, \sqrt{m_2}$):

$$|\hat{u}(\xi,t)| \leq C_M \left(1 \cdot |\hat{u}_0(\xi)| + 1 \cdot |\hat{u}_1(\xi)| + 1 \cdot |\hat{u}_2(\xi)|\right) e^{-c_4 t}$$

$$\text{with } c_4 = \min\left\{\tfrac{1}{\beta}, |\mathrm{Re}(\lambda_{2,3}(\xi_{\nu_1}))|\right\} > 0$$

$$\Downarrow$$

(some integral bounds)

$$\Downarrow$$

$$\left(\int_{\Upsilon_M} |\xi|^{2j} |\hat{u}(\xi,t)|^2 d\xi\right)^{1/2} \leq C(\|\partial_x^j u_0\|_{L^2}^2 + \|\partial_x^j u_1\|_{L^2}^2 + \|\partial_x^j u_2\|_{L^2}^2) e^{-c_4 t}$$

$|\xi|$ large:

$$|\hat{u}(\xi,t)| \leq$$

$$C_H\left(\left(1+\frac{1}{|\xi|}+\frac{1}{|\xi|^2}\right)|\hat{u}_0(\xi)|+\left(\frac{1}{|\xi|}+\frac{1}{|\xi|^2}\right)|\hat{u}_1(\xi)|+\left(\frac{1}{|\xi|^2}+\frac{1}{|\xi|^3}\right)|\hat{u}_2(\xi)|\right)e^{-c_3 t}$$

$$\text{with } c_3=\min\left\{\frac{1}{\beta},\frac{1}{2}\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)\right\}>0$$

$$\Downarrow$$

(some integral bounds)

$$\Downarrow$$

$$\left(\int_{\Upsilon_H}|\xi|^{2j}|\hat{u}(\xi,t)|^2 d\xi\right)^{1/2} \leq C(\|\partial_x^j u_0\|_{L^2}+\|\partial_x^j u_1\|_{L^2}+\|\partial_x^j u_2\|_{L^2})e^{-c_3 t}$$

Joining the previous inequalities we arrive at ...

for $N + j \geq 3$,

$$\begin{aligned} \left\| \partial_x^j u(t) \right\|_{L^2} \quad \leq \quad & C(\|u_0\|_{L^1} + \|u_1\|_{L^1} + \|u_2\|_{L^1})(1+t)^{1/2 - N/4 - j/2} \\ & + C(\|\partial_x^j u_0\|_{L^2} + \|\partial_x^j u_1\|_{L^2} + \|\partial_x^j u_2\|_{L^2}) e^{-ct} \quad \checkmark \end{aligned}$$

with $c = \min\left\{ \frac{1}{\beta}, |\mathrm{Re}(\lambda_{2,3}(\xi_{\nu_1}))| \right\}$

References

- Bounded domain:
  M. P., J. Solà-Morales. *Optimal scalar products in a standard linear viscoelastic model* (Submitted)
- Unbounded domain:
  M. P., B. Said-Houari. *Wellposedness and decay rates for the Cauchy problem of the Moore-Gibson-Thompson equation arising in high intensity ultrasound* (Submitted)

References

- Bounded domain:
  M. P., J. Solà-Morales. *Optimal scalar products in a standard linear viscoelastic model* (Submitted)
- Unbounded domain:
  M. P., B. Said-Houari. *Wellposedness and decay rates for the Cauchy problem of the Moore-Gibson-Thompson equation arising in high intensity ultrasound* (Submitted)

Thank you!