Boundary parameterization in MHD equilibria of Tokamak Plasmas

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Company: Fusion for energy (Dr. Alfredo Portone)



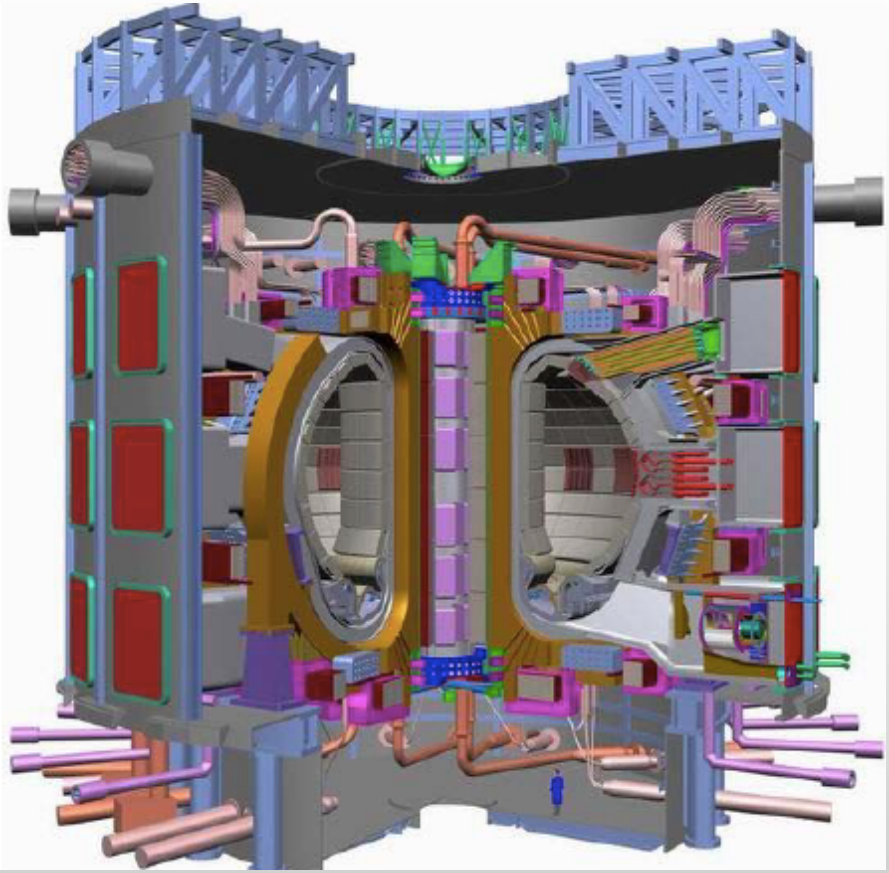
OUTLINE

1. Introduction: company and general problem
2. The bad: plasma boundary parameterization problem
3. The ugly: polynomial interpolation
4. The good: the cubic
5. And now …

INTRODUCTION

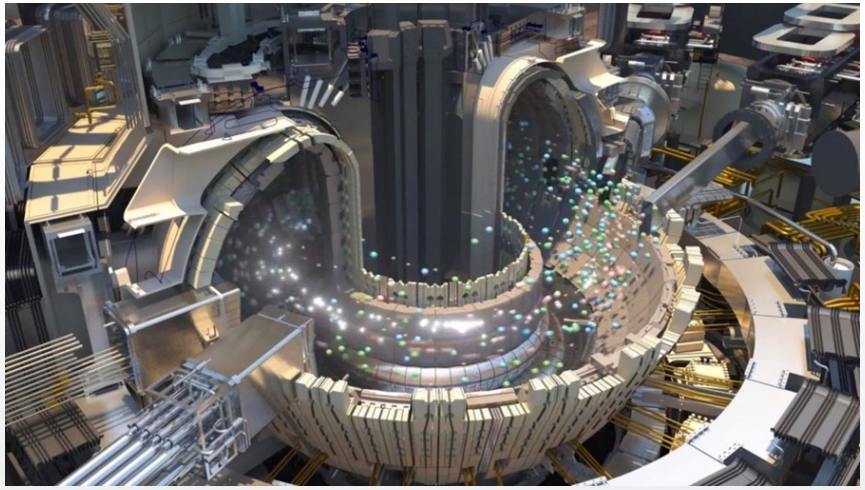
COMPANY AND GENERAL PROBLEM

What is a tokamak?



(Source: www.iterproject.com)

- **Tokamak:** device that confines plasma by magnetic field
  (from Russian: "toroidal chamber with magnetic coils")
- **LARGE TERM GOAL:** generation of energy through nuclear **fusion.**
- Several experimental tokamaks currently in operation in the world.





(Source: www.iterproject.com)

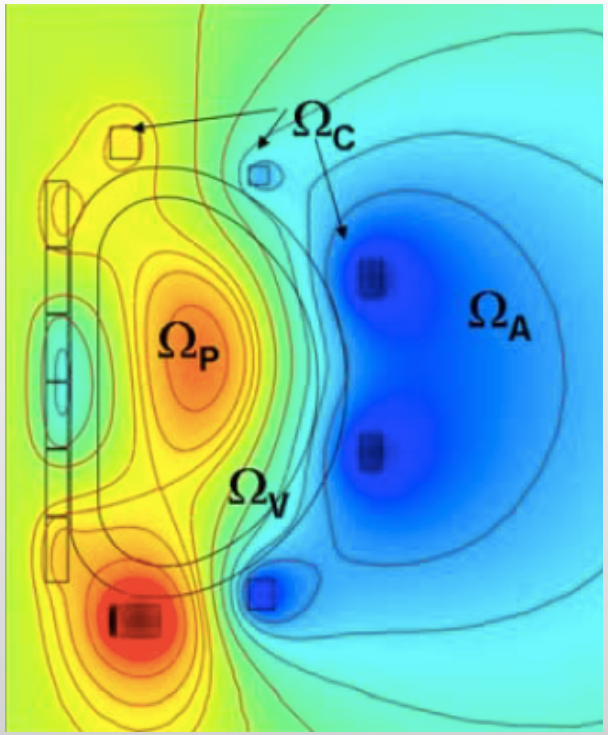
- **ITER project:** world-wide project to design and build the **largest tokamak** (1st stage) and prove the feasibility of the **nuclear fusion energy** (2nd stage) (experimental project).

- **Fusion for Energy (F4E):** European's Union Joint Undertaking for ITER and the Development of Fusion Energy.

- Located in mainly in Barcelona, but also in Cadarache (France) and Garching (Germany).

FUSION FOR ENERGY

Plasma equilibria (MHD, rotational symmetry)



- Grad-Shafranov equation:

$$L\Psi = j \quad \text{with} \quad j = \begin{cases} j(\Psi) & \text{in } \Omega_P \\ j_C & \text{in } \Omega_C \\ 0 & \text{in } \Omega_V \cup \Omega_A \end{cases}$$

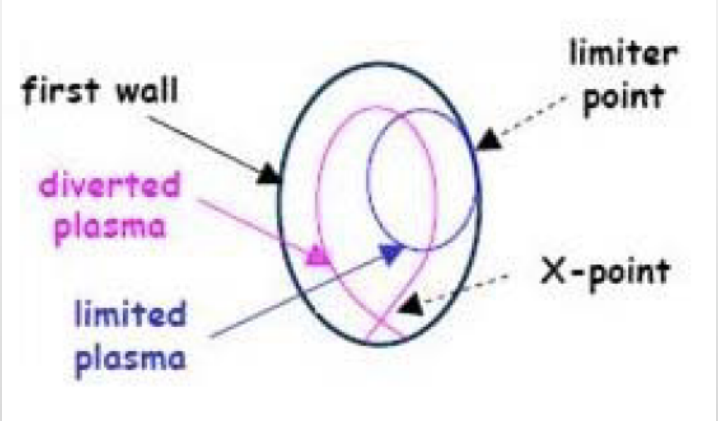
$$L\Psi = -\frac{\partial}{\partial r}\left(\frac{1}{\mu_0 r}\frac{\partial \Psi}{\partial r}\right) - \frac{\partial}{\partial z}\left(\frac{1}{\mu_0 r}\frac{\partial \Psi}{\partial z}\right)$$

- Non-linear. But main problem: Ω_p is unknown (free-boundary problem!).

- Ψ, Ω_p unknowns. Numerical methods.
- From physics,

$$\Psi(\partial\Omega_p) = \text{constant, minimal}$$

- 2 types of critical isolines: limiter and **diverted**



- Iterative numerical scheme:

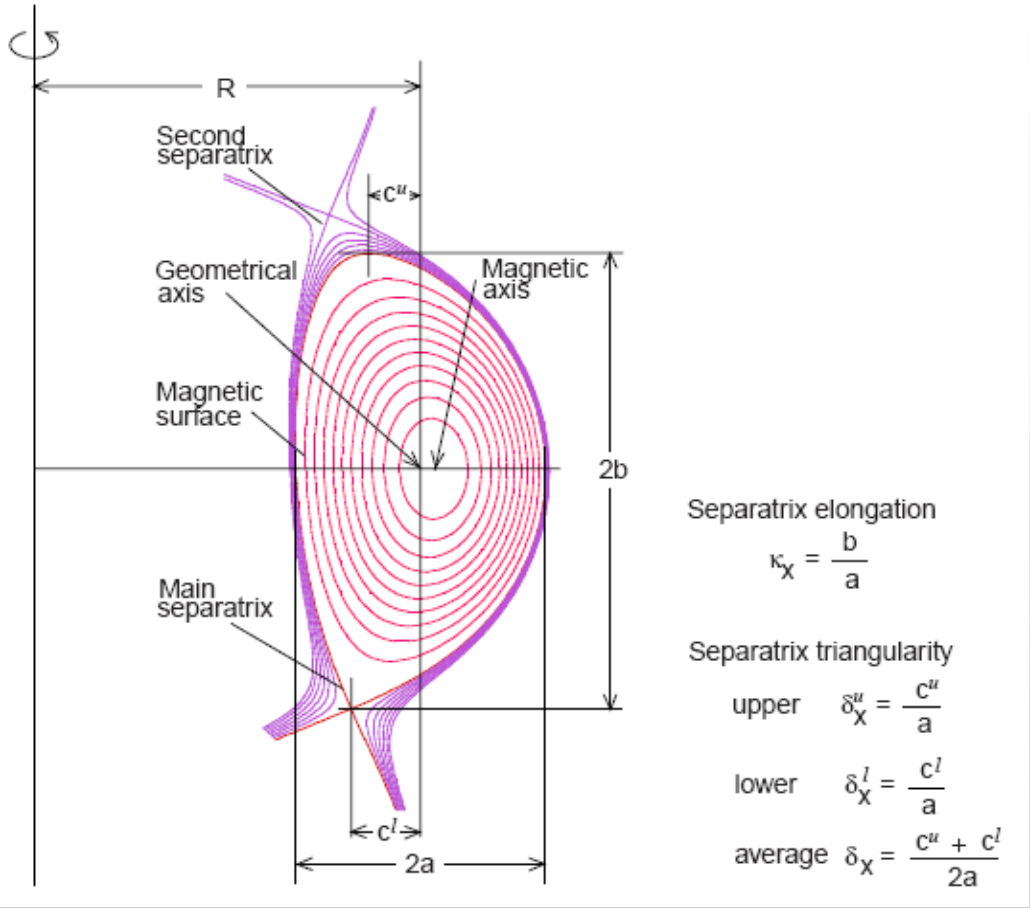
$$\Omega_p\text{: initial guess} \Rightarrow \text{solve G-S} \Rightarrow \Omega_p \Rightarrow \ldots$$

THE BAD

PLASMA BOUNDARY PARAMETERIZATION PROBLEM

(presented in 115th ESGI, Bellaterra, 2016)

PLASMA BOUNDARY PARAMETERIZATION PROBLEM



Limiter plasma:

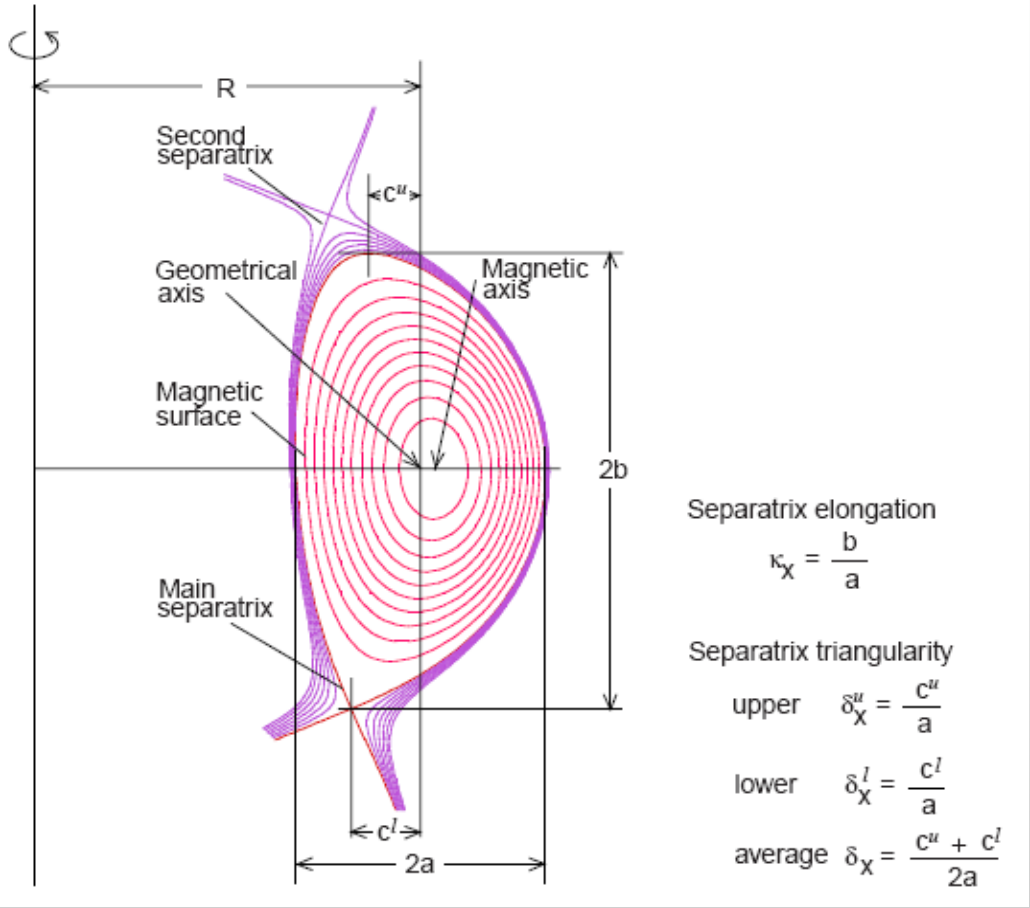
- boundary is a closed magnetic surface and tangent to a material surface

- commonly described by:

$$\begin{cases} r(\theta) = R + a\cos(\theta + \delta\sin(\theta)) \\ z(\theta) = k\sin(\theta) \end{cases}$$

R, a: major, minor radius
k: elongation
δ: triangularity
$\theta \in [0, 2\pi)$: poloidal angle

PLASMA BOUNDARY PARAMETERIZATION PROBLEM



Diverted plasma:

- bounded by a flux surface that is NOT closed neither in contact with any material surface, but it contains saddle point(s) at which the boundary has cusp(s) of 90°

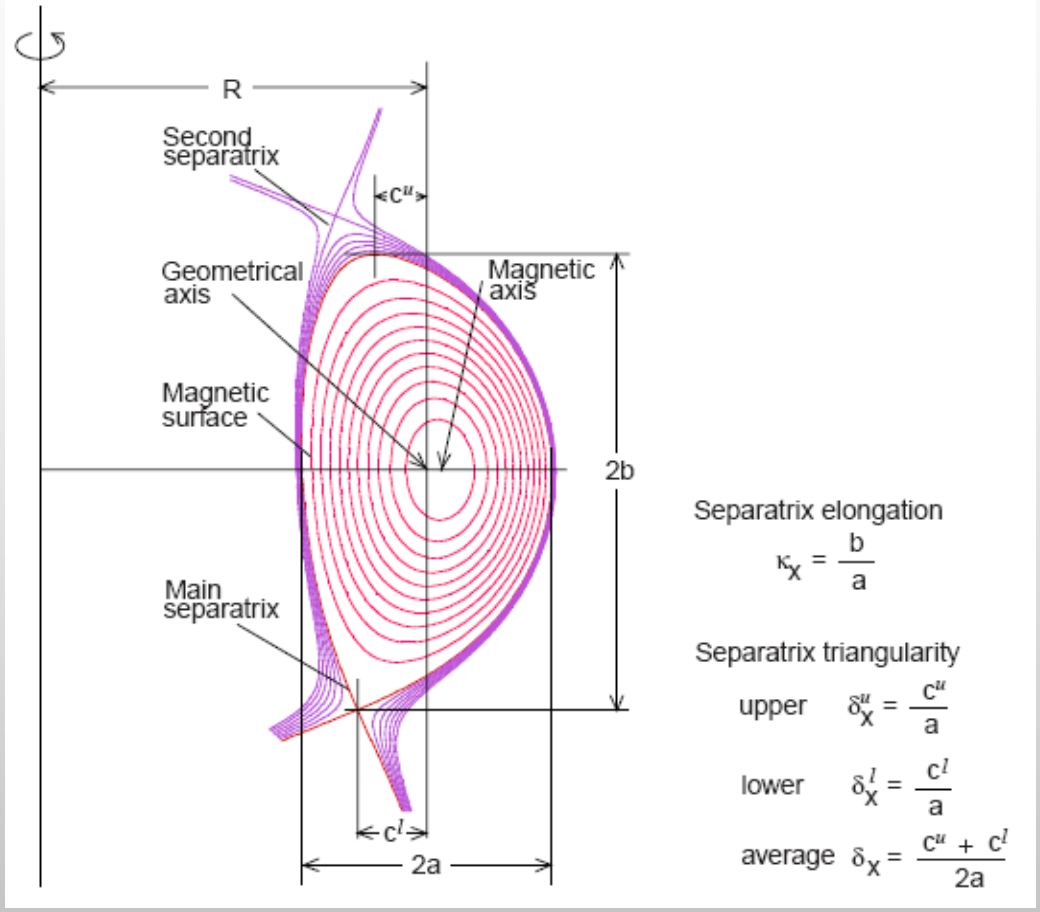
- for such type of boundary (which is the most commonly used in burning plasmas) **the parameterization above is NO longer valid.**

F4E PROBLEM:

Given R, a, k, δ find a "simple" **parameterization**

of a diverted plasma boundary

… that is, find a curve whose only constraints in the (R,Z) plane are :



1. **Tangent at the "vertical" direction** (Z-axis) at (Ri,0) and (Ro,0) (min. and max. radius of the curve, *specified by the user*).

2. Must contain a **saddle point** at (Rx,Zx) where the branches cross at 90°. *(point specified by the user)*

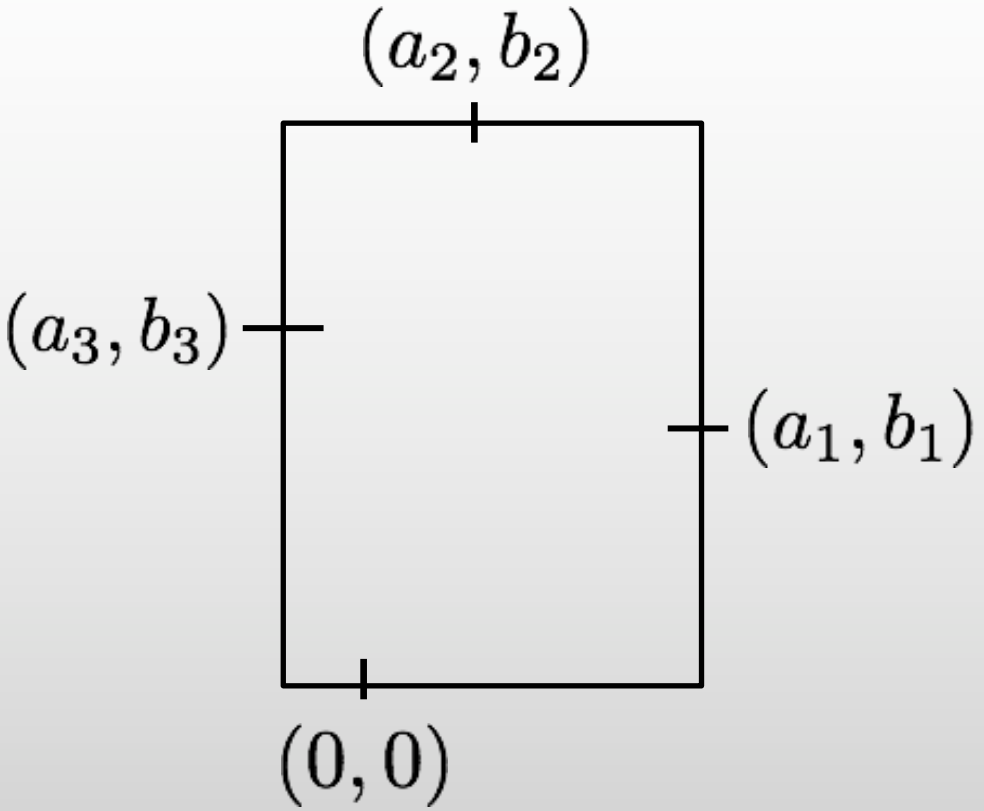
3. Not necessarily symmetric respect to Z=0 plane

REMARKS:

- Geometric problem (not physical)

- **What they said:** parameterization
  What they meant: "simple formula"/ characterization of the **geometric** shape of the curve

Before any parameterization …



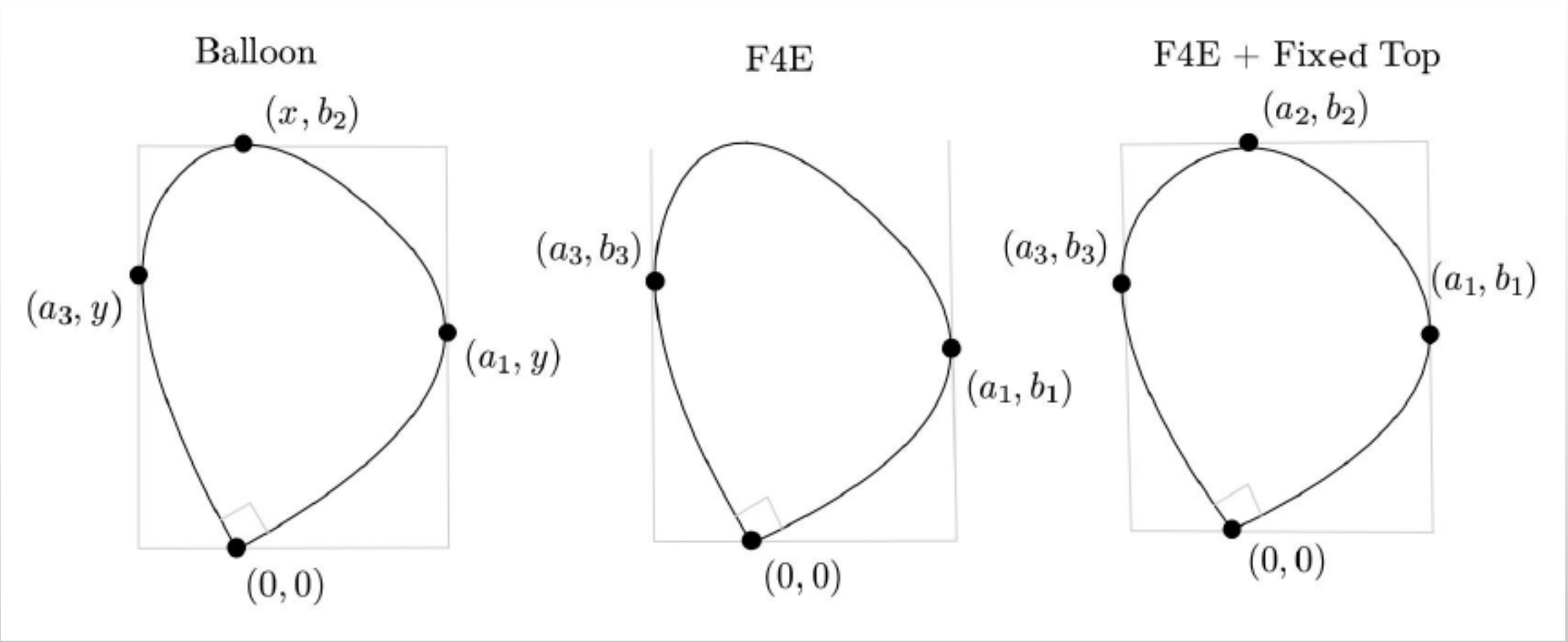
- Translation into **(x,y) plane**
- Cross point at **(0,0)**
- Curve **vertices**

  (instead of geometric parameters):

$$
\begin{aligned}
a_1 &= R + a, \\
a_2 &= 2R - 2a\delta, \\
a_3 &= R - a, \\
b_2 &= 2ak.
\end{aligned}
$$

Remark: when cross points, additional parameters needed (e.g. slopes at cross point)

We tried several problems:



THE UGLY:

POLYNOMIAL INTERPOLATION

Parametric polynomial interpolation

$$(x(t), y(t)) = (x_0 + x_1 t + x_2 t^2 + \ldots + x_m t^m, y_0 + y_1 t + y_2 t^2 + \ldots + y_n t^n), \ t \in [0, 1]$$

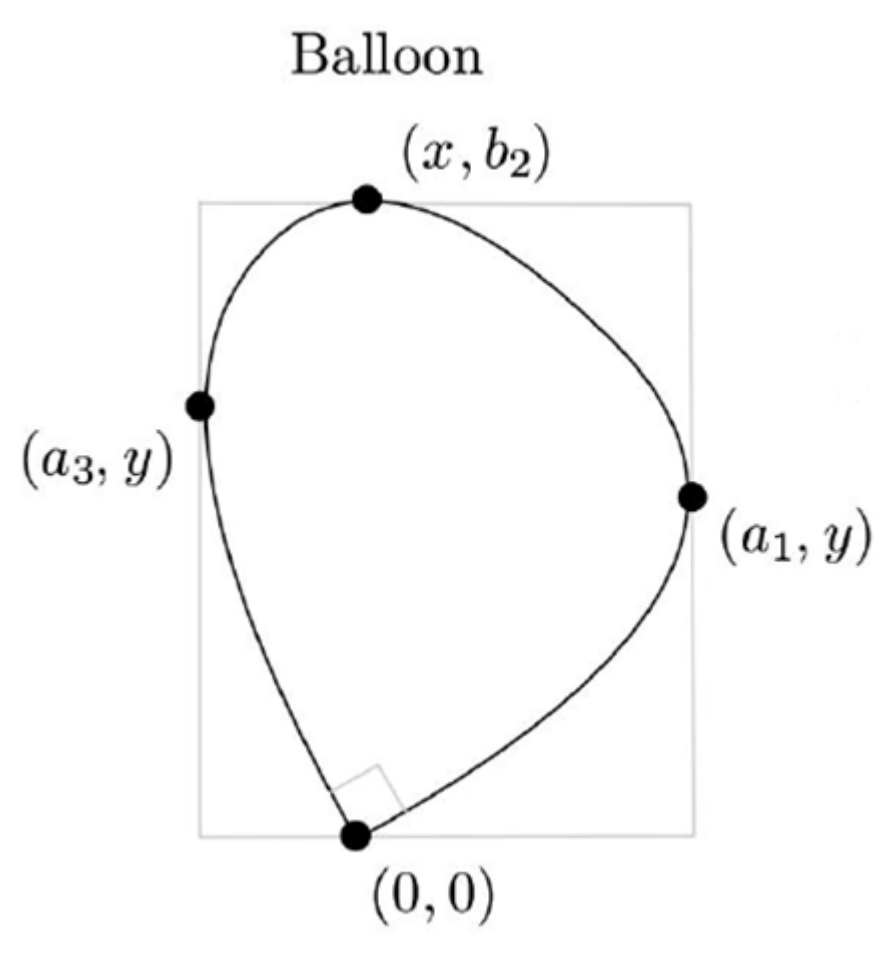
Type of constraints:

- $(0, 0)$ at $t = 0$ and $t = 1$
- (a_i, b_i) at $t = t_i$
- Vertical or horizontal tangencies at $t = t_i$
- m_0, m_1 right and left slopes at $t = 0$ and $t = 1$



LINEAR SYSTEM OF EQUATIONS in the coefficients

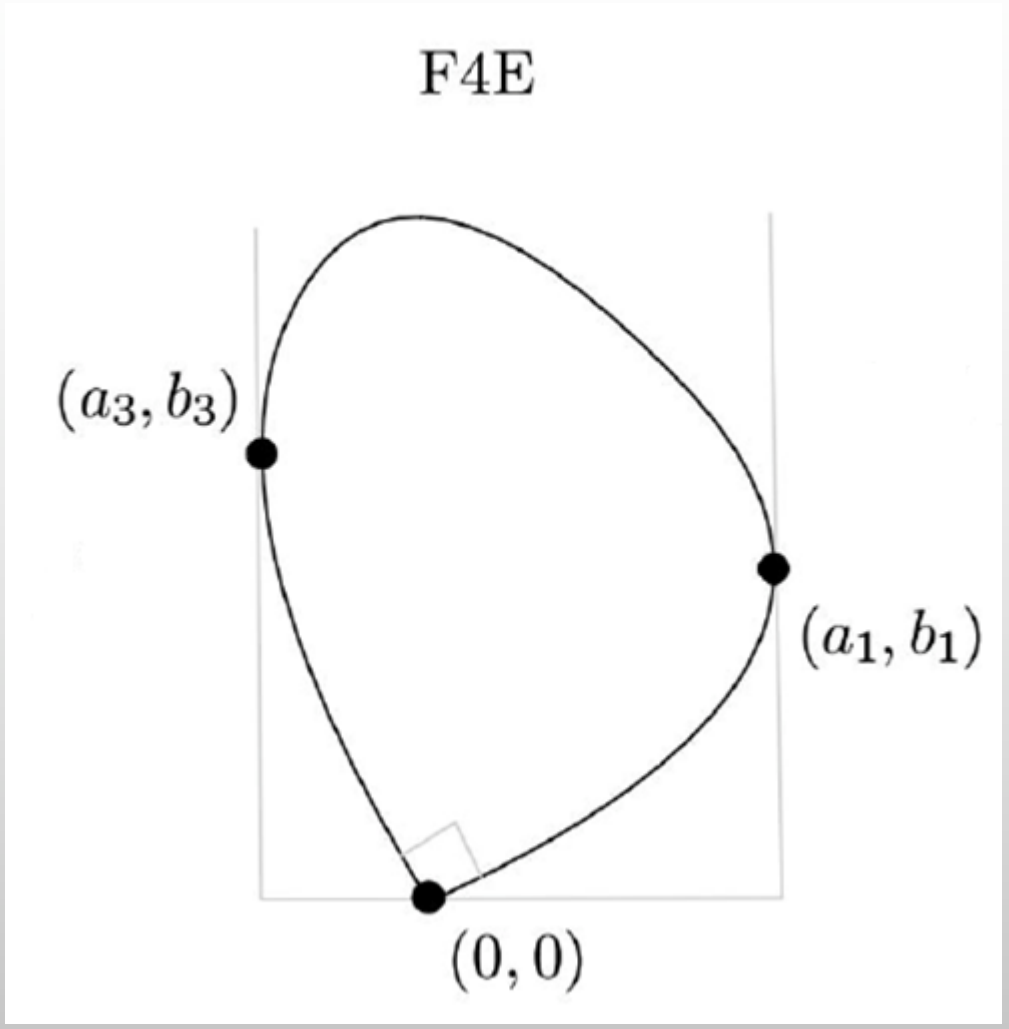
The balloon problem



- $(0, 0)$ cross point.
- Vertical tangencies at $x = a_1$ and $x = a_3$, horizontal at $y = b_2$.
- m_0, m_1 fixed right and left slopes at $(0, 0)$. Right angle condition if $m_0 \cdot m_1 = -1$.

Interpolation by x(t), y(t) polynomials (degree 5 and 3)

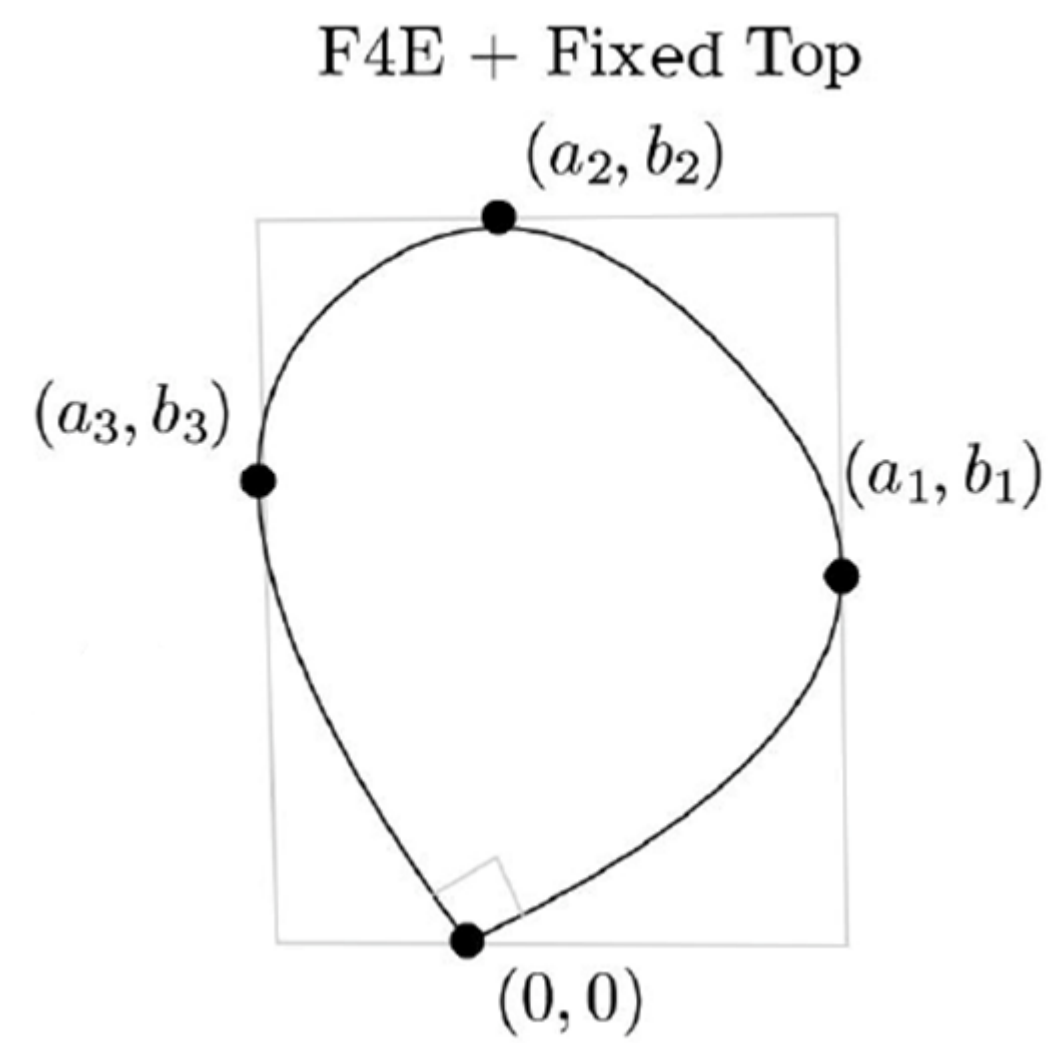
The F4E problem



- $(0, 0)$ cross point.
- Passes through (a_1, b_1) and (a_3, b_3).
- Vertical tangencies at the previous points.
- m_0, m_1 fixed right and left slopes at $(0, 0)$. Right angle condition if $m_0 \cdot m_1 = -1$.

Interpolation by x(t), y(t) polynomials (degree 5 and 5)

The F4E problem + fixed top and height

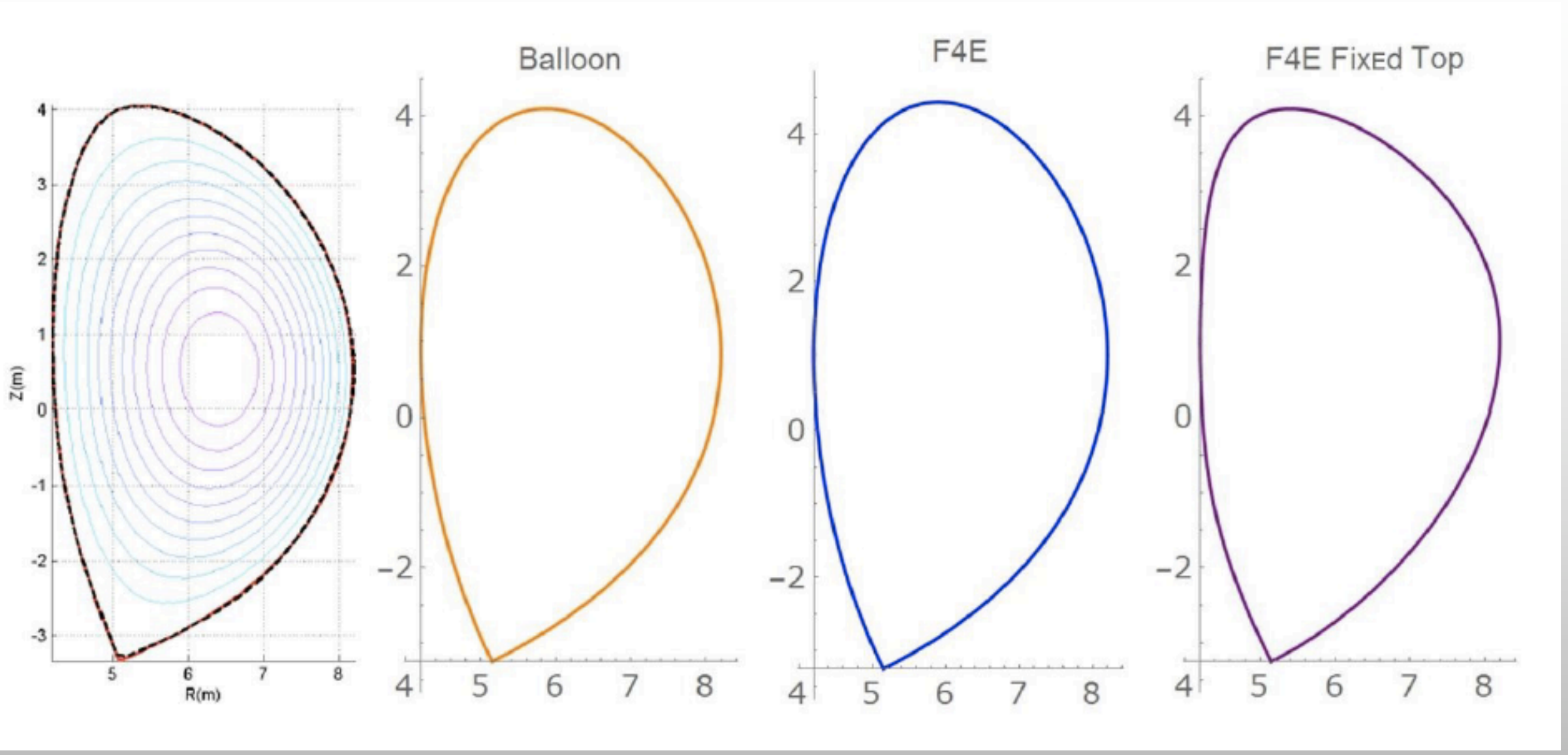


As F4E problem and ...

- passes through (a_2, b_2) and horizontal tangent there.

Interpolation by x(t), y(t) polynomials (degree 6 and 7)

Numerical results



REMARKS:

- Can obtain many interpolating curves by tunning time parameters ($0<t1<t2<t3<1$) (adjusted to fit the target curve).
- Degrees of the polynomials depend on the imposed conditions (possible Runge phenomenon).
- Very sensitive to geometric parameters.
- Other possibilities for interpolation …

  (maybe piecewise parameterization)

THE GOOD:

THE CUBIC

MOTIVATION: level curves of a hamiltonian with a **cubic potential energy**

GOAL: find an implicit representation of the curve as

$$H(x,y) = 0$$

(**cubic polynomial**). Then, convert it into a parameterization.

(taking into account the amount of constraints, it is worth considering a **cubic**)

GENERAL FORM OF A CUBIC

$$H(x,y) = \alpha_{00} + \alpha_{10}x + \alpha_{01}y + \alpha_{20}x^2 + \alpha_{11}xy + \alpha_{02}y^2 + \alpha_{30}x^3 + \alpha_{21}x^2y + \alpha_{12}xy^2 + \alpha_{03}y^3$$

If (0,0) saddle point and after normalization (0-level curve) …

$$H(x,y) = x^2 + \alpha_{11}xy + \alpha_{02}y^2 + \alpha_{30}x^3 + \alpha_{21}x^2y + \alpha_{12}xy^2 + \alpha_{03}y^3$$

Remark: up to **6 constraints** (appart from (0,0) cross point) can be imposed.

If < 6: free coefficients (used as tunning ones)

If > 6: we cannot use the cubic approach

And then, the parameterization …

- Sheaf of straight lines through (0,0):

$$y = t\,x \text{ } (t\text{: slope w.r.t. horizontal axis})$$

- 3 intersection points with H(x,y) = 0:
  - (0,0) twice
  - The third one gives the parameterization

$$x(t) = -\left(\frac{\alpha_{02}t^2 + \alpha_{11}t + 1}{\alpha_{03}t^3 + \alpha_{12}t^2 + \alpha_{21}t + \alpha_{30}}\right), \quad y(t) = t\,x(t) \;\;,\; t \in [m_1, m_0]$$

Type of constraints:

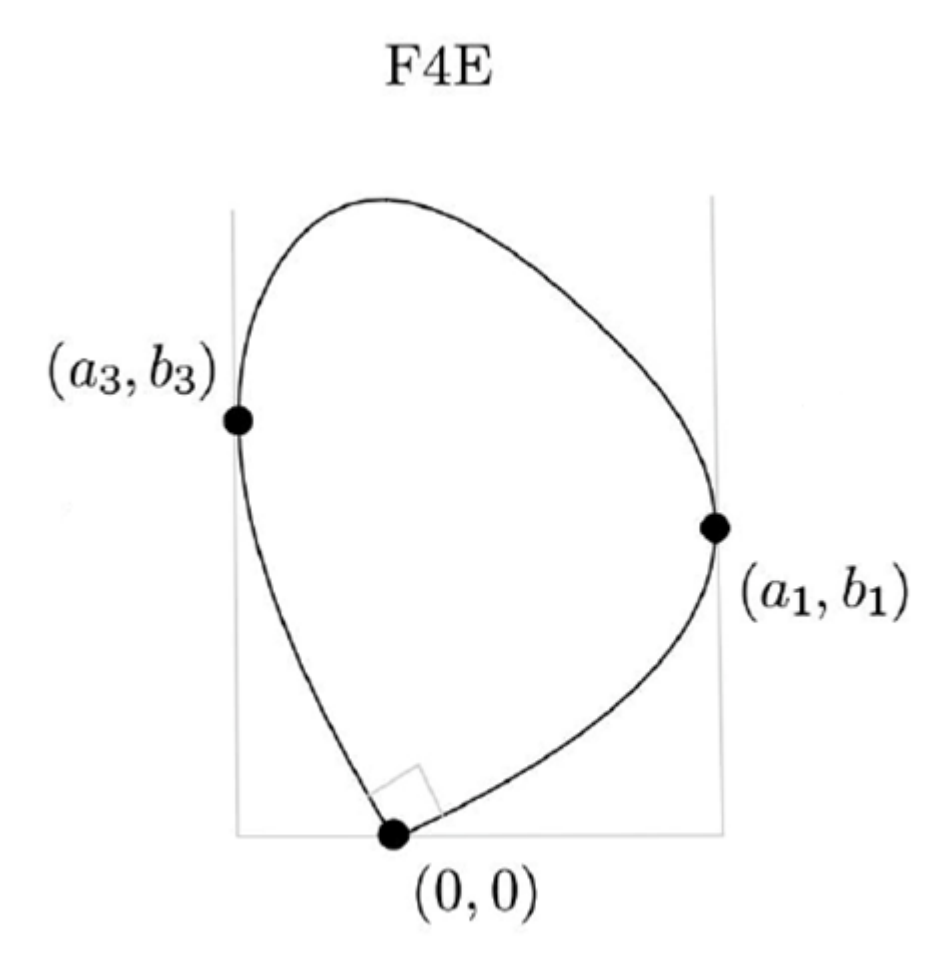
- $H(a_i, b_i) = 0$ (linear condition in α_{ij})
- Tangency conditions.
  - at a point (linear in α_{ij}):
    $H(a_i, b_i) = H_y(a_i, b_i) = 0$ (vertical tangent)
    $H(a_i, b_i) = H_x(a_i, b_i) = 0$ (horizontal tangent)
  - at a line:
    $H(a_i, y) = H_y(a_i, y) = 0$ (vertical tangent)
    $H(x, b_i) = H_x(x, b_i) = 0$ (horizontal tangent)

    But x, y? $\Rightarrow$ use discriminant to impose it $\Rightarrow$ non-linear in α_{ij}
- Slope conditions:

$$\alpha_{11} = -\left(\frac{m_0 + m_1}{m_0 m_1}\right), \quad \alpha_{02} = \frac{1}{m_0 m_1}$$

(right angle condition gives $\alpha_{02} = -1$)

The F4E problem

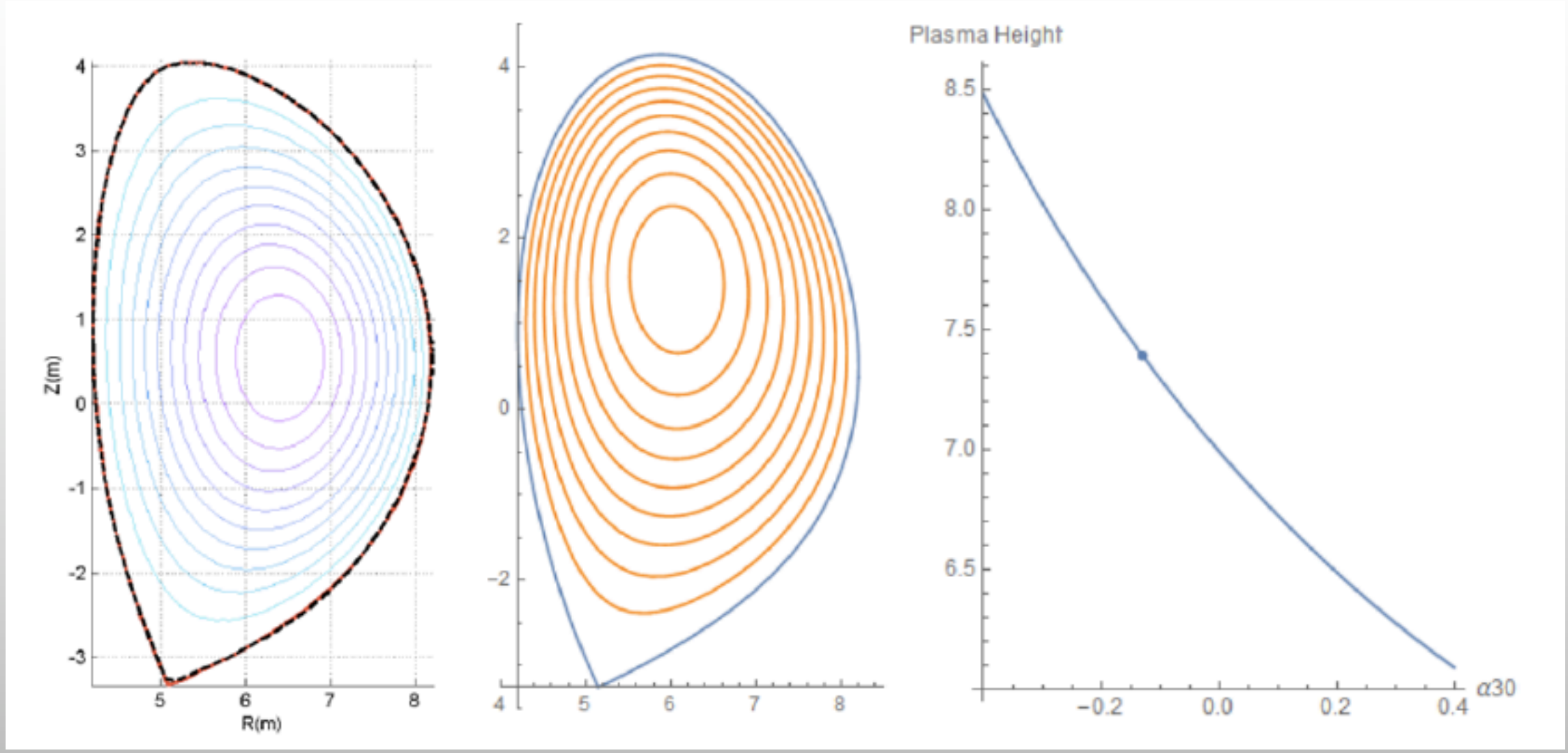


- $(0, 0)$ cross point.
- Passes through (a_1, b_1) and (a_3, b_3).
- Vertical tangencies at the previous points.
- m_0, m_1 fixed right and left slopes at $(0, 0)$. Right angle condition if $m_0 \cdot m_1 = -1$.

H(x,y) = 0, with **linear** relations between the coefficients (red: tunning parameter):

$$H(x,y) = x^2 + \alpha_{11}xy - y^2 + \alpha_{30}x^3 + \alpha_{21}x^2y + \alpha_{12}xy^2 + \alpha_{03}y^3$$

Numerical results (right: tunning parameter)



AND NOW …

- Company satisfied with results
- Problem and the needs of the company much more defined now
- We have continued the work:
  - quartic approach: more constraints allowed, but no parameterization
  - piecewise parameterization: Bézier curves, (generalized) splines, …
  - complex analysis ?
- Collaboration goes on …

SOME BIBLIOGRAPHY …

- **"MHD equilibria of Tokamak plasmas".** C. Barril , A. Haro, M. Pellicer, J. Solà-Morales. *Reports of 115th ESGI (to appear)*

- "**Noncircular, finite aspect ratio, local equilibrium model".** R. L. Miller, et al. *Physics of Plasmas, 5 (1998)*
- **"Magnet system optimization in tokamak engineering".** A. Portone. *Decision and Control (CDC), 2012*
- **"Computation of MHD Equilibrium of Tokamak Plasma".** T. Takeda, S. Tokuda. *J. Comput. Phys. (1991).*

THANK YOU FOR YOUR ATTENTION !!!

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