Optimal decay rates of the Moore-Gibson-Thompson equation.

Marta Pellicer (Universitat de Girona, Catalunya)

Work in collaboration with:

J. Solà-Morales (Universitat Politècnica de Catalunya) (bounded problem)
B. Said-Houari (Sharjah University, UAE) (problem in $\mathbb{R}^N$)

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Outline

1. Introduction and modeling
   - Acoustic approach: MGT equation
   - Mechanical approach: SLV model
2. MGT in a bounded domain
   - Description of the spectrum of A
   - Main result 1: new scalar product and normality of A
   - Main result 2: optimal exponential decay
3. MGT in $\mathbb{R}^N$
   - Energy method in the Fourier space
   - The eigenvalues expansion method (solutions decay)

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Acoustic approach: MGT equation

- Propagation of **sound**
- Acoustic velocity potential in viscous thermally relaxing fluids
- Medical and industrial application of **high intensity focused ultrasound** (lithotripsy, thermotherapy, ultrasound cleaning, . . .)

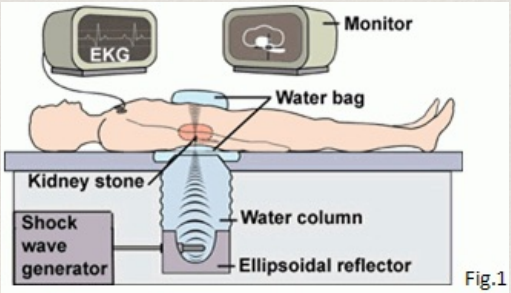
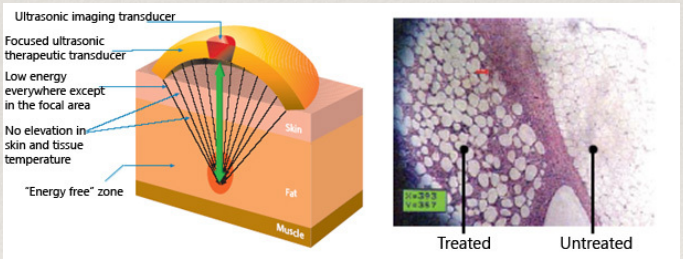


Fig.1



- Fourier law for heat conduction ($q = -K\nabla\theta$):

  $\Rightarrow$ **Kuznetsov equation** (classic nonlinear Acoustics)

$$u_{tt} - c^2\Delta u - b\Delta u_t = \frac{\partial}{\partial t}\left(\frac{1}{c^2}\frac{B}{2A}(u_t)^2 + |\nabla u|^2\right)$$

  Problem: infinite speed of propagation !!!
  (not good for previous applications)

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- Alternative $\Rightarrow$ Maxwell-Cattaneo law for heat conduction:

$$\tau q_t + q = -K\nabla\theta$$

  (τ: relaxation time of heat flux $\Rightarrow$ thermal inertia therm):

  $\Rightarrow$ **Jordan-Moore-Gibson-Thompson equation**:

$$\tau u_{ttt} + u_{tt} - c^2\Delta u - b\Delta u_t = \frac{\partial}{\partial t}\left(\frac{1}{c^2}\frac{B}{2A}(u_t)^2 + |\nabla u|^2\right)$$

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  Linear part $\Rightarrow$ **Moore-Gibson-Thompson equation**:

$$\boxed{\tau u_{ttt} + u_{tt} - c^2\Delta u - b\Delta u_t = 0}$$

Mechanical approach: SLV model

Deformation of a **viscoelastic material**
(properties: viscous fluidity + elastic solidity)





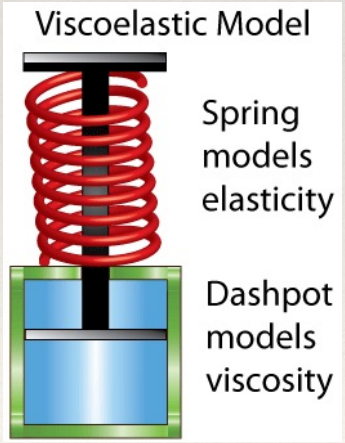
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Rheological approach (1D):



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- relation between σ (**stress**, force) and ε (**strain**, deformation)
- combination of

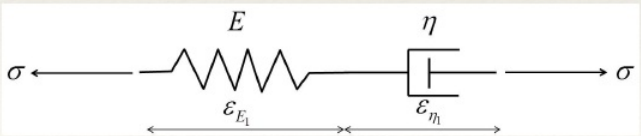
spring (elasticity) and **dashpot** (viscosity)

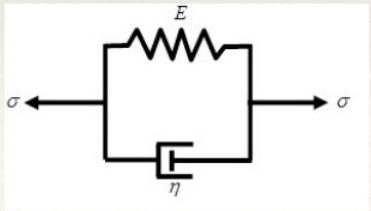
$$\sigma = E\,\varepsilon \qquad \sigma = \eta\,\varepsilon'$$

in **series** or in **parallel**

$$\varepsilon = \varepsilon_e + \varepsilon_v \qquad \varepsilon = \varepsilon_e = \varepsilon_v$$

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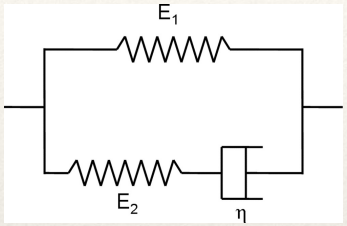




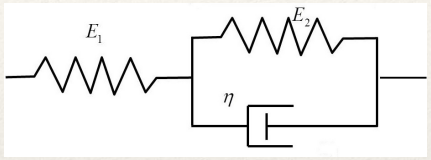
- mechanical tests: creeping and stress relaxation
- simplest models:
  - Kelvin-Voigt (in **parallel**, wave equation with **strong** damping)
  - Maxwel (in **series**, wave equation with **weak** damping).

The Standard Linear Viscoelastic model

- a more realistic approach
- the most simple exhibiting creeping and stress relaxation
- 2 rheological approaches:



spring + Maxwell in **parallel**



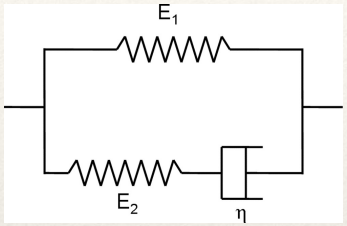
spring + Kelvin-Voigt in **series**

Same constitutive equation: $\sigma + \tau\sigma' = E\left(\varepsilon + \beta\varepsilon'\right)$

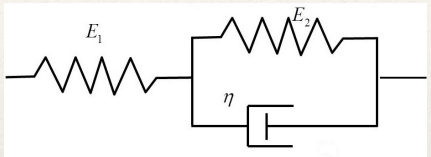
(different relation of the coefficients with the physical parameters)

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$\Downarrow$

The continuous model (SLV model):

$$\tau u_{ttt} + u_{tt} - c^2\left(u_{xx} + \beta u_{txx}\right) = 0,\ x \in [0, L],\ t \geq 0$$

where $u(x,t) =$ **deformation**, $\tau, \beta > 0$.

Some references

- (A) B. Kaltenbacher, I. Lasiecka, R. Marchand (Control Cybernet., 2011)
- (A) B. Kaltenbacher, I. Lasiecka, M. K Pospieszalska (M^3ASc, 2012)
- (A) R. Marchand, T. McDevitt, R. Triggiani (M^3ASc, 2012)

- (M) G. C. Gorain, (Proc. Indian Acad. Sci. (Math. Sci.), 2010)
- (M) M.S. Alves, C. Buriol, M.V. Ferreira, J.E. Muñoz Rivera, M. Sepúlveda, O. Vera (JMAA, 2013)
- (M) J.A. Conejero, C. Lizama, F. Ródenas (App. Math. Inf. Sciences, 2015)

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The problem

Consider

$$\tau u_{ttt} + u_{tt} - c^2 \Delta \left(u + \beta u_t \right) = 0, \ x \in \Omega \subset \mathbb{R}^N$$

with homogeneous Dirichlet b.c.

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with homogeneous Dirichlet b.c.

In general:

$$\tau\, u_{ttt} + u_{tt} + L (u + \beta u_t) = 0, \ x \in \Omega \subset \mathbb{R}^N$$

- Ω bounded, regular domain
- L self-adjoint, > 0, compact resolvent operator in H (Hilbert) *(eigenvalues:* $0 < \mu_1 \leq \mu_2 \leq \ldots \to \infty$*, semi-simple eigenfunctions:* ϕ_n *orthonormal)*
- several possible functional spaces $\mathcal{H}_i$ *(see [M-McD-T, 2012])*

Important remarks and known results (previous references)

$$\tau\, u_{ttt} + u_{tt} - c^2 \Delta u - c^2 \beta \Delta u_t = 0,\ x \in \Omega \subset \mathbb{R}^N$$

- Although linear, this equation displays a variety of physical behaviors depending on the parameter values (**non-existence**, **existence but chaotic**, **exponential decay**)
- $\tau > 0 \Rightarrow$ hyperbolic type equation

Important remarks and known results (previous references)

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Well-posedness (β)

- $\beta = 0$ (and L unbounded): $\nexists$ C^0-semigroup
- $\beta > 0$: $\exists$ C^0-group.

Decay (τ/β)

- $\tau < \beta$ **(dissipative case)**: **exponentially stable group**
- $\tau = \beta$ (conservative case): constant energy
- $\tau > \beta$: chaotic behaviour

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Remark: third parameter c^2, "hidden" in the abstract form through the spectrum of $L = -c^2\Delta$ (important role in the eigenvalues related conditions, $c^2\mu_n$)

Description of the spectrum of A

We write it as a first order system:

$$\frac{dU}{dt} = AU \ , \ U \in D(A), \ \text{with } AU = \begin{pmatrix} v \\ w \\ -\frac{1}{\tau} L(u + \beta v) - \frac{1}{\tau} w \end{pmatrix}$$

The eigenvalues:

Characteristic equation:

$$\tau\lambda^3 + \lambda^2 + \beta\mu_n\lambda + \mu_n = 0$$

Cubic equation $\Rightarrow$ 3 solutions for each n:

$$\lambda_1^n, \lambda_2^n, \lambda_3^n$$

Type depending on the sign of Cardano's discriminant, with zeroes

$$m_1, \ m_2$$

In general, **1 real** sequence λ_1^n and **2 nonreal** ones, $\lambda_2^n = \overline{\lambda_3^n}$

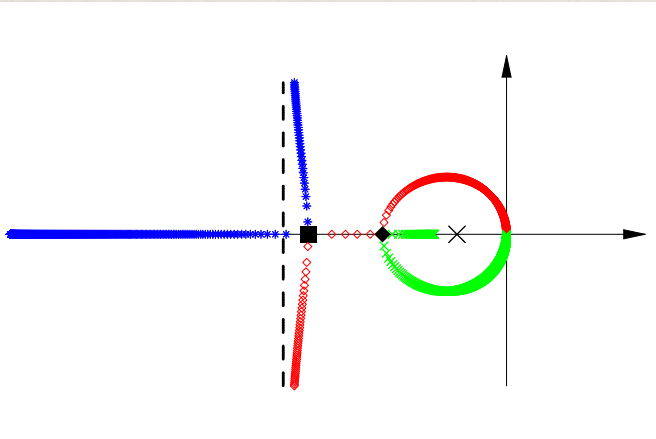
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EXCEPT ... (not considered in [MMcDT, 2012])

If $\tau/\beta < 1/9$ (**overdamping**) and $\mu_n \in [m_1, m_2]$ (finite number!):

3 real eigenvalues

simple if $\mu_n \in (m_1, m_2)$, 1 simple and 2 **double** if $\mu_n = m_1$ or m_2, **triple** if $\tau/\beta = 1/9$ and $\mu_n = m_1 (= m_2)$.



Spectrum when $\tau/\beta < 1/9$

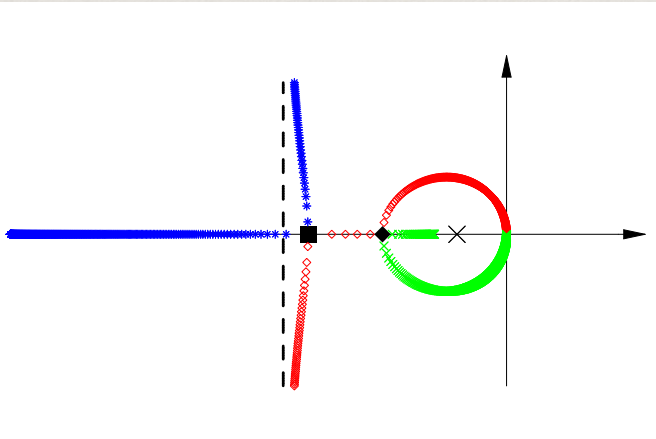
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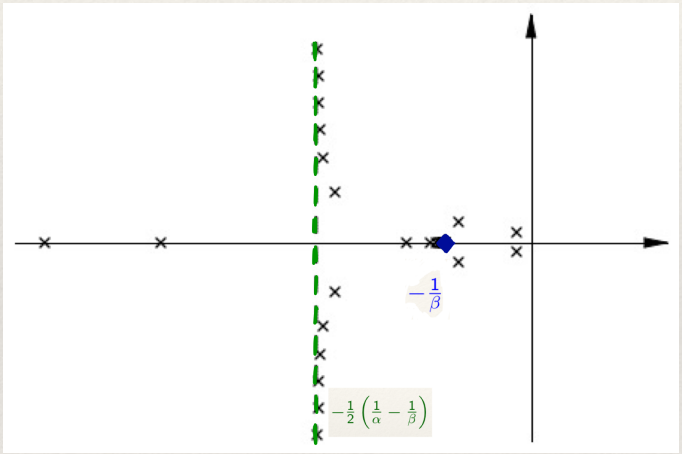


Spectrum when $\tau/\beta < 1/9$

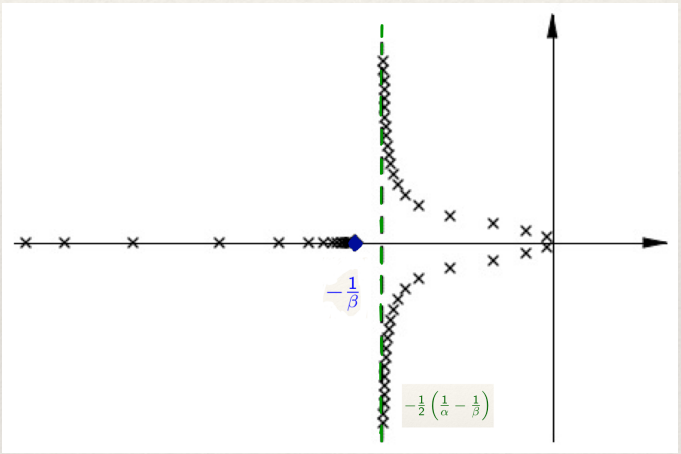
- In the **discrete case**, it can happen or not (depending on the parameters) (but it **DOES** happen as $\tau \to 0$!!)
- In the **continuous case (unbounded domain)** it **DOES** happen ([S-H,MP, AMO 2017]).
- This will be tricky and important in next results

They fulfill:

- $\lambda_k^n \in \mathbb{R} \Rightarrow \lambda_k^n \nearrow -\frac{1}{\beta}$ **(essential spectrum)**
- $\lambda_k^n \in \mathbb{C} \setminus \mathbb{R} \Rightarrow 0 > \mathrm{Re}(\lambda_k^n) \searrow -\frac{1}{2}\left(\frac{1}{\tau} - \frac{1}{\beta}\right)$
- $-\frac{1}{2}\left(\frac{1}{\tau} - \frac{1}{\beta}\right) < -\frac{1}{\beta}$ if $\frac{\tau}{\beta} < \frac{1}{3}$ (resp. $=$, $>$)



$\frac{\tau}{\beta} < \frac{1}{3}$

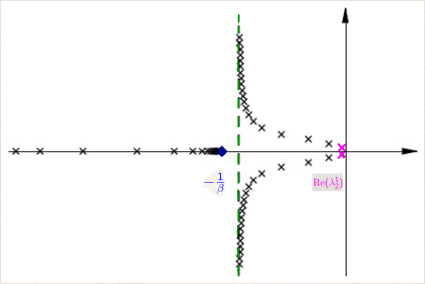


$\frac{\tau}{\beta} > \frac{1}{3}$

Dominant spectrum: two possibilities

- $\{\lambda_2^1, \lambda_3^1\}$ (the first two nonreal eigenvalues)
- $-\frac{1}{\beta}$ (essential spectrum) (**overdamping**: no dominant oscillations)

$$\text{If } \frac{\tau}{\beta} > \frac{1}{3} \Rightarrow \sigma_{max} = \mathrm{Re}(\lambda_2^1):$$

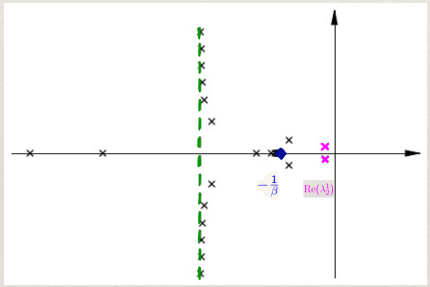


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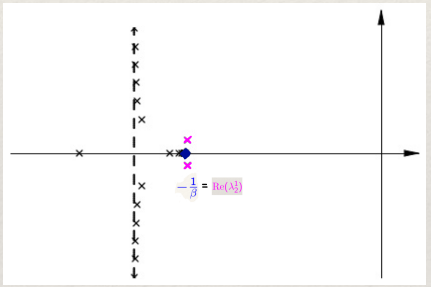
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$$\text{If } \frac{\tau}{\beta} < \frac{1}{3} \Rightarrow \text{different possibilities}$$

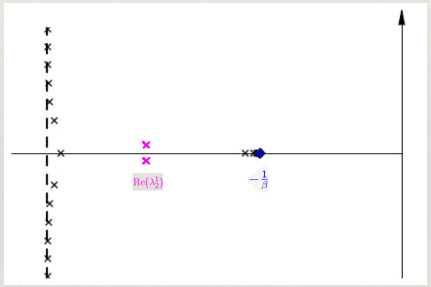
(still not explicit condition: solve the characteristic equation for $\mu_n = \mu_1$)



$\sigma_{max} = \mathrm{Re}(\lambda_2^1)$



$\sigma_{max} = \mathrm{Re}(\lambda_2^1) = -\frac{1}{\beta}$



$\sigma_{max} = -\frac{1}{\beta}$

Main result 1: new scalar product and normality of A

Theorem 1

- If $\mu_n \neq m_1,\ m_2$:
  - we construct a **new and explicit scalar product** $\langle \cdot, \cdot \rangle_G$ where A is a **normal** operator
  - the eigenfunctions are **orthonormal and complete** with G

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Remarks

- A normal if $AA^* = A^*A$
  ($\Rightarrow$ diagonable, orthonormal eigenfunctions)

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- A normal is the best we can expect
  (A is not self-adjoint as it has non-real eigenvalues)
- Main difficulty: proving the equivalence of the new scalar product and the natural one

The idea ...

If A normal in $\langle , \rangle_G \Rightarrow$ A diagonable $\Rightarrow$
$\Psi_1^n, \Psi_2^n, \Psi_3^n$ orthonormal eigenfunctions in this norm.

So, it is natural to define

$$G = \begin{pmatrix} G_1 & 0 & \cdots & 0 & \cdots \\ 0 & G_2 & \cdots & 0 & \cdots \\ \vdots & \vdots & \ddots & \vdots & \ddots \\ 0 & 0 & \cdots & G_n & \cdots \\ \vdots & \vdots & \ddots & \vdots & \ddots \end{pmatrix}$$

where $\langle u, v \rangle_{G_n} = \langle C_n^{-1} u, C_n^{-1} v \rangle_{\mathbb{R}^3}$ (with $C_n = \mathrm{col}(\Psi_1^n, \Psi_2^n, \Psi_3^n)$).

$$\Rightarrow G_n = (\overline{C_n^{-1}})^T C_n^{-1}.$$

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Important (and tricky):

$\langle , \rangle_{G_n} \sim \langle , \rangle$ **uniformly in** $n \Rightarrow$ G equivalent to the natural product

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Remark

Only possible when non multiple eigenvalues !!!
(in which case A **not normal** because not diagonable)

Example in $\mathcal{H} = H_0^1(\Omega) \times H_0^1(\Omega) \times L^2(\Omega)$*:*

$$O_n = \begin{pmatrix} \mu_n & 0 & 0 \\ 0 & \mu_n & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$G_n = \begin{pmatrix} \frac{2\tau\beta^2+3\tau+\beta}{2\tau\beta^2}\mu_n + o(\mu_n) & \frac{\tau+\beta}{2\tau\beta}\mu_n + o(\mu_n) & \frac{4\tau^2\beta^2+\beta^2+3\tau^2}{4\tau\beta^3}\cdot 1 + o(1) \\ \frac{\tau+\beta}{2\tau\beta}\mu_n + o(\mu_n) & \frac{\tau+\beta}{2\tau}\mu_n + o(\mu_n) & \frac{(\tau+\beta)^2}{4\tau\beta^2}\cdot 1 + o(1) \\ \frac{4\tau^2\beta^2+\beta^2+3\tau^2}{4\tau\beta^3}\cdot 1 + o(1) & \frac{(\tau+\beta)^2}{4\tau\beta^2}\cdot 1 + o(1) & \frac{\tau+\beta}{2\beta}\cdot 1 + o(1) \end{pmatrix}$$

Main result 2: optimal exponential decay

Theorem 2

- If A normal in $\langle \cdot, \cdot \rangle_G$:

$$\|U(t)\|_G \leq e^{\sigma_{max} t} \|U(0)\|_G$$

$\sigma_{max} = \max\{\mathrm{Re}(\lambda_2^1), -\frac{1}{\beta}\} < 0$, **dominant spectrum**.
Optimal exponent.

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Remarks

In [MMcDT 2012]: exponential decay with the growth bound

$$\inf\{\omega \in \mathbb{R}; \|e^{At}\| \leq M_\omega e^{\omega t} \ \forall t \geq 0\} \quad \textit{(usual norm)}$$

We improve it: $M_\omega = 1$, and the infimum is a **minimum** (also when σ_{max} is the essential spectrum)

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If A normal in $\langle , \rangle_G$ ...

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$$\Downarrow$$

In this basis:

$$\|U(t)\|_G^2 = \sum_{n,j} |d_j^n|^2 e^{2Re(\lambda_j^n)t} \leq \sum_{n,j} |d_j^n|^2 e^{2\sigma_{max} t} \ (t > 0). \ \checkmark$$

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- Optimality:
  - Clear if $\sigma_{max} = \mathrm{Re}(\lambda_2^1)$ ✓
  - Also if $\sigma_{max} = -\frac{1}{\beta}$, as we have $\lambda_1^n \to -\frac{1}{\beta}$ ✓

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- Multiple eigenvalues are not a problem because **they are not dominant** $\Rightarrow$ we can essentially do the same as before
- $A = \begin{pmatrix} A_0 & 0 \\ 0 & A_1 \end{pmatrix}$ (A_1 the part of the multiple eigenvalues)
  For A_0 the same as before applies:

$$\|e^{A_0 t}\|_{G_0} \leq e^{\sigma_{max}(A_0)\, t} \leq e^{\sigma_{max}(A)\, t}$$

  For A_1 we can define $\langle \cdot, \cdot \rangle_{G_{1,\varepsilon}}$ such that (for a certain $\varepsilon > 0$)

$$\|e^{A_1 t}\|_{G_{1,\varepsilon}} \leq e^{(\sigma_{max}(A_1)+\varepsilon)\, t} \leq e^{-\frac{1}{\beta}\, t} \leq e^{\sigma_{max}(A)\, t}$$

  $\Rightarrow$ G' defined as the orthogonal extension of G_0, G_1 ✓

Outline

1. Introduction and modeling
   - Acoustic approach: MGT equation
   - Mechanical approach: SLV model
2. MGT in a bounded domain
   - Description of the spectrum of A
   - Main result 1: new scalar product and normality of A
   - Main result 2: optimal exponential decay
3. MGT in $\mathbb{R}^N$
   - Energy method in the Fourier space
   - The eigenvalues expansion method (solutions decay)

The problem in $\mathbb{R}^N$

We consider

$$\tau u_{ttt} + u_{tt} - \Delta u - \beta \Delta u_t = 0 \text{ in } \mathbb{R}^N$$

with $0 < \tau < \beta$ (we take $c = 1$ without loss of generality)

What we do?

- well-posedness (C_0-semigroup of contractions) ✓
  (not done before in $\mathbb{R}^N$)
- decay rate results

The decay results (I)

Theorem (decay of the energy):

$$\|\nabla^j V(t)\|_{L^2(\mathbb{R}^N)} \leq C(1+t)^{-N/4-j/2}\|V_0\|_{L^1(\mathbb{R}^N)} + Ce^{-ct}\|\nabla^j V_0\|_{L^2(\mathbb{R}^N)}$$

where $V = (u_t + \tau u_{tt}, \nabla(u + \tau u_t), \nabla u_t)$, $c > 0$.

Idea of the proof: Lyapunov functional method in the Fourier space + Plancherel Theorem.

Remark: the heuristics suggest the decay is optimal, but this is not a bound for the solution.

The decay results (II)

Theorem (decay of the solutions):

For $N + j \geq 3$,

$$\begin{aligned} \left\|\nabla^j u(t)\right\|_{L^2} \quad \leq \quad & C(\|u_0\|_{L^1} + \|u_1\|_{L^1} + \|u_2\|_{L^1})(1+t)^{1/2-N/4-j/2} \\ & + C(\|\nabla^j u_0\|_{L^2} + \|\nabla^j u_1\|_{L^2} + \|\nabla^j u_2\|_{L^2})e^{ct} \end{aligned}$$

with $c = \max\left\{-\frac{1}{\beta}, \operatorname{Re}(\lambda_{2,3}(\xi_{\nu_1}))\right\} < 0$

Idea of the proof: eigenvalues expansion method, using the previous description of the spectrum.

Decay (low frequency eigenvalues): for all $N \geq 3$ and all $j \geq 0$ (for $N = 1, 2$ decay only if $j > 3/2,\ 1/2$, but similar result gives decay $(1+t)^{1-N/4-j/2}$ for other $j > 0$). Does not seem optimal.

Regularity (high frequency eigenvalues): same as the initial conditions (no regularity-loss type).

Idea: eigenvalues expansion method

Based on the **asymptotic behaviour of the eigenvalues** in the Fourier space.

Observe that same description of the spectrum as in the bounded domain, but $|\xi|$ ($\sim \sqrt{\mu_n}$) being a continuous parameter.

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Observe that same description of the spectrum as in the bounded domain, but $|\xi|$ ($\sim \sqrt{\mu_n}$) being a continuous parameter.

How?

- Plancherel Theorem:

$$\left\|\nabla^j u\left(t\right)\right\|_{L^2}^2 = \int_{\mathbb{R}^N} |\xi|^{2j} |\hat{u}\left(\xi, t\right)|^2 d\xi$$

- Split the previous integral in Υ_L, Υ_M, Υ_H
- And, roughly speaking ... ODE in the Fourier space for each ξ

$$\hat{u}(\xi, t) = C_1(\xi) e^{\lambda_1(\xi) t} + e^{\mathrm{Re}(\lambda_2(\xi)) t} \left[C_2(\xi) \cos(\mathrm{Im}(\lambda_2(\xi)) t) + C_3(\xi) \sin(\mathrm{Im}(\lambda_2(\xi)) t)\right]$$

($C_i(\xi)$ using initial conditions and asymptotic behaviour of the eigenvalues in each part)

Asymptotic behaviour of the eigenvalues in each part:

- $\Upsilon_L = \{|\xi| < \nu_1 << 1\} \Rightarrow$ **decay** (as $\hat{u} \sim e^{\mathrm{Re}(\lambda(\xi))t}$)

$$\begin{cases} \lambda_1(\xi) \sim -\dfrac{1}{\tau}, \\ \lambda_2(\xi) \sim i|\xi| - \dfrac{1}{2}(\beta - \tau)|\xi|^2, \\ \lambda_3(\xi) \sim -i|\xi| - \dfrac{1}{2}(\beta - \tau)|\xi|^2 \end{cases}$$

- $\Upsilon_M = \{\nu_1 \leq |\xi| \leq \nu_2\}$ (multiple eigenvalues not a problem):

$$\mathrm{Re}(\lambda^n_{1,2,3}(\xi)) < \max\left\{-\frac{1}{\beta}, \mathrm{Re}(\lambda_{2,3}(\xi_{\nu_1}))\right\} < 0$$

- $\Upsilon_H = \{|\xi| > \nu_2 >> 1\} \Rightarrow$ **regularity** (as $\hat{u} \sim |\xi|^m e^{-c|\xi|^{-\hat{c}}t}$)

$$\begin{cases} \lambda_1(\xi) \sim -\dfrac{1}{\beta}, \\ \lambda_2(\xi) \sim -\dfrac{\beta - \tau}{2\beta\tau} + i|\xi|\sqrt{\dfrac{\beta}{\tau}}, \\ \lambda_3(\xi) \sim -\dfrac{\beta - \tau}{2\beta\tau} - i|\xi|\sqrt{\dfrac{\beta}{\tau}} \end{cases}$$

Imposing initial conditions (and taking the dominant terms) ...

$|\xi|$ **small:**

$$
\begin{aligned}
|\hat{u}(\xi,t)| \quad \leq \quad & C_L \left(|\xi|^2 |\hat{u}_0| + |\xi|^2 |\hat{u}_1| + |\hat{u}_2| \right) e^{-\frac{1}{\tau} t} \\
& + C_L \left(|\hat{u}_0| + |\xi|^2 |\hat{u}_1| + |\hat{u}_2| \right) e^{-\frac{\beta-\tau}{2} |\xi|^2 t} \cos(|\xi| t) \\
& + C_L \left(|\xi| |\hat{u}_0| + \frac{1}{|\xi|} |\hat{u}_1| + \frac{1}{|\xi|} |\hat{u}_2| \right) e^{-\frac{\beta-\tau}{2} |\xi|^2 t} \sin(|\xi| t)
\end{aligned}
$$

$$\Downarrow$$

(some integral bounds for $N + j \geq 3$ or for all $N \geq 1$)

$$\Downarrow$$

$$
\left(\int_{\Upsilon_L} |\xi|^{2j} |\hat{u}(\xi,t)|^2 d\xi \right)^{1/2} \leq C(\|u_0\|_{L^1}^2 + \|u_1\|_{L^1}^2 + \|u_2\|_{L^1}^2)(1+t)^{\frac{1}{2} - \frac{N}{4} - \frac{j}{2}}
$$

$|\xi|$ **medium** (and $|\xi| \neq \sqrt{m_1}, \sqrt{m_2}$):

$$|\hat{u}(\xi,t)| \leq C_M \left(1 \cdot |\hat{u}_0(\xi)| + 1 \cdot |\hat{u}_1(\xi)| + 1 \cdot |\hat{u}_2(\xi)|\right) e^{c_1 t}$$

$$\text{with } c_1 = \max\left\{-\tfrac{1}{\beta}, \mathrm{Re}(\lambda_{2,3}(\xi_{\nu_1}))\right\} < 0$$

$$\Downarrow$$

(some integral bounds)

$$\Downarrow$$

$$\left(\int_{\Upsilon_M} |\xi|^{2j} |\hat{u}(\xi,t)|^2 d\xi\right)^{1/2} \leq C(\|\nabla^j u_0\|_{L^2}^2 + \|\nabla^j u_1\|_{L^2}^2 + \|\nabla^j u_2\|_{L^2}^2) e^{c_1 t}$$

$|\xi|$ **large:**

$$|\hat{u}(\xi,t)| \leq$$

$$C_H\left(\left(1+\frac{1}{|\xi|}+\frac{1}{|\xi|^2}\right)|\hat{u}_0(\xi)|+\left(\frac{1}{|\xi|}+\frac{1}{|\xi|^2}\right)|\hat{u}_1(\xi)|+\left(\frac{1}{|\xi|^2}+\frac{1}{|\xi|^3}\right)|\hat{u}_2(\xi)|\right)e^{c_2 t}$$

$$\text{with } c_2=\max\left\{-\tfrac{1}{\beta},\tfrac{1}{2}\left(\tfrac{1}{\beta}-\tfrac{1}{\tau}\right)\right\}<0$$

$$\Downarrow$$

(some integral bounds)

$$\Downarrow$$

$$\left(\int_{\Upsilon_H}|\xi|^{2j}|\hat{u}\left(\xi,t\right)|^2 d\xi\right)^{1/2}\leq C(\|\nabla^j u_0\|_{L^2}+\|\nabla^j u_1\|_{L^2}+\|\nabla^j u_2\|_{L^2})e^{c_2 t}$$

Joining the previous inequalities we arrive at ...

for $N + j \geq 3$,

$$\begin{aligned} \left\|\nabla^j u(t)\right\|_{L^2} \quad \leq \quad & C(\|u_0\|_{L^1} + \|u_1\|_{L^1} + \|u_2\|_{L^1})(1+t)^{1/2 - N/4 - j/2} \\ & + C(\|\nabla^j u_0\|_{L^2} + \|\nabla^j u_1\|_{L^2} + \|\nabla^j u_2\|_{L^2}) e^{ct} \quad \checkmark \end{aligned}$$

with $c = \max\left\{-\frac{1}{\beta}, \mathrm{Re}(\lambda_{2,3}(\xi_{\nu_1}))\right\} < 0$.

References

- Bounded domain:
  M.P., J. Solà-Morales.
  Optimal scalar products in the Moore-Gibson-Thompson equation. *EECT (2019).*

- Unbounded domain:
  M.P., B. Said-Houari.
  Wellposedness and decay rates for the Cauchy problem of the Moore-Gibson-Thompson equation arising in high intensity ultrasound. *AMO (2017).*

Universitat de Girona

Thank you!

http://imae.udg.edu/∼martap/