Voltage drop in on-chip power distribution networks

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3.1 Introduction

The complexity of the different digital circuits with different functions that are becoming integrated in current chips is growing every day. These different components can be found in the market as closed objects, which would be too expensive to re-design in each individual case. These consume electric power at significant rates, which are also quite variable in time. These time variations in some parts of the chip induce undesirable fluctuations in the power supply to the other parts, producing delays and malfunctions in the logic gates. For these reasons, control of maximum voltage drops in the different parts of the chip is one of the most important issues in its design.

Mathematically speaking, we identify the chip with a rectangle $Q = [0, a] \times [0, b]$, and to calculate the voltage drop $u(x, y)$ on it one has to solve the Poisson-like partial differential equation

$$\nabla \cdot \left(\frac{1}{R(x, y)} \nabla u \right) = J(x, y), \tag{3.1}$$

where the sheet resistance function $R(x, y)$ and the current density $J(x, y)$ are known functions. These two functions are usually approximated by functions that are constant across the components of the chip, which are usually of a rectangular form and are usually known as *blocks*. The equation (3.1) has to be solved together with some boundary conditions, which will depend on the type of chip.

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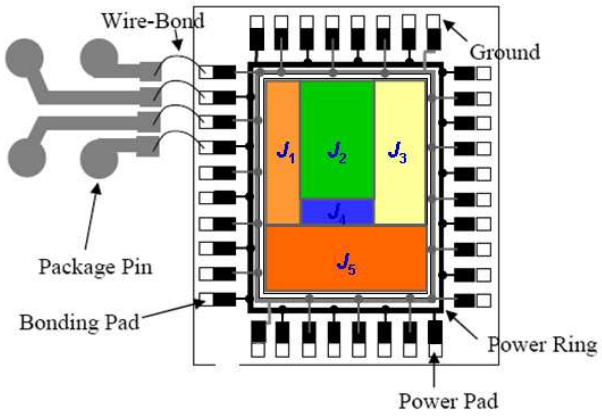


Figure 3.1: Sketch of a wire-bond chip.

In wire-bond chips, the connection to a zero potential conductor is done at the boundaries of Q (see Figure 3.1), so the equation (3.1) has to be solved in Q together with the boundary condition

$$u(x, y) = 0 \quad \text{on the boundary of } Q. \tag{3.2}$$

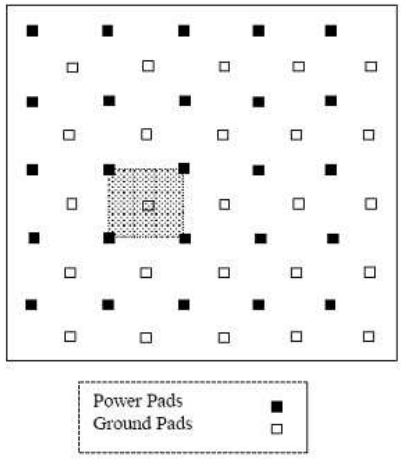


Figure 3.2: Sketch of a flip-chip.

In the flip-chip case, the connection to ground is done through a net of *pads* that are represented by the white small squares in Figure 3.2. In that

case, the boundaries of Q appear as insulating walls, in the sense that there is no external voltage gradient at these points and hence no current flowing through the boundary. If we call P the set of all these pads, we have to solve

$$\begin{cases} \nabla \cdot \left(\dfrac{1}{R(x,y)} \nabla u \right) = J(x,y) & \text{in } Q \setminus P, \\ \dfrac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \partial Q, \\ u = 0 & \text{on } \partial P. \end{cases} \tag{3.3}$$

The problems (3.1), (3.2) or (3.3) can be quite easily solved with numerical or semi-numerical methods, and we will review these methods later on. However, the designer of chip circuits needs to have online clues about the location of the dangerous zones of large voltage drops before completing the full design. And when these dangerous situations appear, the designer needs to have procedures to decrease these undesirable effects along the way. Completing the full design, realizing that the specifications are not fulfilled and restarting the design again would be a waste of time.

The problem that we have discussed, then, is to find all possible tools that could contribute to the design of a software for *online* support of the design process of these kinds of chips in future work.

This report is based on contributions from the discussions held at the Grups d'Estudi*, which included the following members: Maria Aguareles, Jordi Blasco, Fernando Martínez, Marta Pellicer, Antonio Rodríguez-Ferran and Joan Solà-Morales.*

3.2 Fourier series in the wire-bond chip

Double sine Fourier series are very well adapted to the solution of Poisson equations in rectangles with zero boundary conditions (see [1], for example). Let us suppose for the moment that the function $R(x,y)$ in (3.1) is constant, say $R(x,y) \equiv R_0 > 0$. Then one has to solve

$$\begin{cases} \nabla^2 u = R_0 J(x,y) & \text{in } Q = [0,a] \times [0,b], \\ u = 0 & \text{on } \partial Q. \end{cases} \tag{3.4}$$

This problem has a straightforward solution:

$$u(x,y) = \sum_{m,n \geq 1} E_{m,n} \sin\left(\frac{m\pi}{a} x\right) \sin\left(\frac{n\pi}{b} y\right), \tag{3.5}$$

where

$$E_{m,n} = \frac{4R_0}{\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)\pi^2 ab} \int_0^a \int_0^b J(x,y) \sin\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right)\, dxdy.$$

Now let us suppose that the function $J(x,y)$ is piecewise constant, with values $J_1, J_2, \ldots, J_p$ on subrectangles $Q_1, Q_2, \ldots, Q_p$ that cover Q (and we still keep $R(x,y) \equiv R_0$ everywhere on Q). Then an observation that could be useful is to note that we can write the solution of (3.4) as

$$u(x,y) = \sum_{k=1}^{p} u^k(x,y),$$

where each u^k solves (3.4) with a right-hand side that is $J_k\varphi^k(x,y)$, where $\varphi^k(x,y)$ is constantly equal to 1 in Q_k and equal to zero on $Q \setminus Q_k$.

One first property of this decomposition process is that one can calculate the functions $u^k(x,y)$ successively, and this means that, along a design process, every time a new component is added the designer can get an idea of the effect of the new component into the previously designed parts. We believe that this property is quite relevant and it also holds, with the natural variations, in the case of variable $R(x,y)$ and also for the flip-chip configurations.

But a second property of this decomposition in the present case is that the Fourier series expression of the partial solutions $u^k(x,y)$ can be explicitly given (see Figure 3.3). If Q_k is the rectangle of dimensions $a_k \times b_k$ centered at the point (x_k, y_k), then the solution $u^k(x,y)$ is given by (3.5), where the values of the $E_{m,n}$ are explicitly

$$E^k_{m,n} = \frac{16R_0J_k}{\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)\pi^4 mn} \sin\left(\frac{m\pi}{a}x_k\right) \sin\left(\frac{a_k m\pi}{2a}\right) \sin\left(\frac{n\pi}{b}y_k\right) \sin\left(\frac{b_k n\pi}{2b}\right).$$

The double Fourier series method can only be applied to (3.1) and (3.2) when the coefficient function $R(x,y)$ is constant. Nevertheless, we present an idea, that should perhaps be further developed in a future work, for the use of the Fourier series when the function $R(x,y)$ is no longer constant throughout the whole chip, but takes different constant values R_k in each Q_k block, which is indeed a much more realistic situation. In that case we could start by making an estimate of a reasonable possible average value R_0 and proceeding as before, but now calculating a correction term $v(x,y)$ that will be small if the sheet resistances at each block are similar, that is, where $\varepsilon_k = R_k - R_0$ are small.

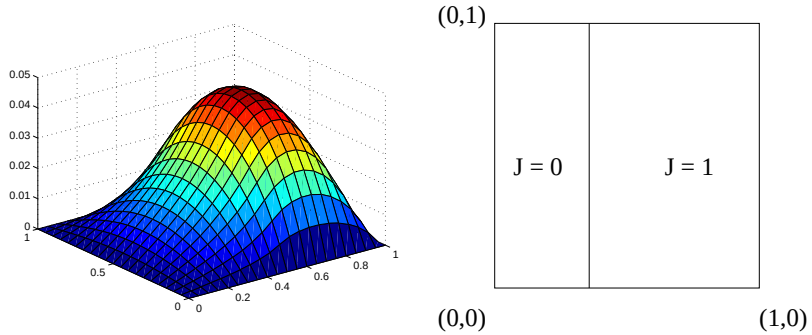


Figure 3.3: Solution to equation (3.4) with two blocks: the left one with $J_1 = 0$ and the second one with $J_1 = 1$.

We first look for an approximation to this correction term of the form $v(x, y) = \sum_{k=1}^{n} \varepsilon_k v^k(x, y)$. If we plug $u + v$ into (3.1) and (3.2), use that u satisfies (3.4), and drop the quadratic terms in the ε_k, we obtain the following equation for v^k:

$$\begin{cases} \nabla^2 v^k = \dfrac{1}{R_0} \nabla \cdot (\varphi^k \nabla u) & \text{in } Q = [0, a] \times [0, b], \\ v^k = 0 & \text{on } \partial Q, \end{cases} \tag{3.6}$$

and this problem can also be solved with double Fourier sine series, as in (3.4) and (3.5).

As we have already pointed out, the double Fourier sine series can only be used if the Laplacian has constant coefficients. However, there is a second way to solve the problem with variable coefficients that also converts the non-constant coefficients problem into infinitely further problems with constant coefficients. The method consists in writing the equation (3.1) in terms of a fixed point equation to be solved by iteration:

$$\nabla \cdot \left(\frac{1}{R_0} \nabla u^{(\ell+1)} \right) + \nabla \cdot \left(\left(\frac{1}{R(x, y)} - \frac{1}{R_0} \right) \nabla u^{(\ell)} \right) = J(x, y),$$

and if $R(x, y)$ is sufficiently close to R_0 it will converge to the actual solution. Each step of this iteration can be done with the method of the Fourier series. Observe that if one starts with $u^{(0)} \equiv 0$, then $u^{(1)}$ will be an approximate solution of the same form as the one in (3.4).

3.3 Asymptotic analysis in the flip-chip

We now tackle the problem of determining the voltage drop in a flip-chip package. This problem was successfully studied by Shakery and Meindl in [5] for the case where both R and J are constant. They considered the two cases represented in Figure 3.4, where the pads are at a distance a from each other and they are either circles of radius ε or squares of sides 2ε. Using the periodicity and symmetry properties of the geometry, one may focus on solving the problems that are represented in Figure 3.5, where one has taken the center of the pad as the origin of coordinates, and then the maximum voltage drop is given by the value $u(a/2, a/2)$.

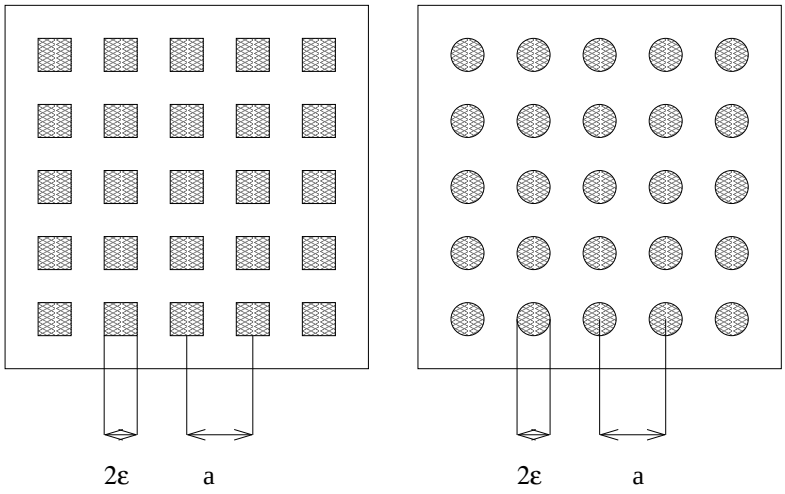


Figure 3.4: Flip-chip with squared or circle pads.

The formula obtained in [5, formula (31)] for the approximate value of $u(a/2, a/2)$ is

$$\frac{RJa^2}{2\pi} \ln \varepsilon - \frac{RJa^2}{2\pi} \ln(0.387a) \tag{3.7}$$

when the pads are circular, and

$$\frac{RJa^2}{2\pi} \ln \left(\frac{0.5903}{0.5} \varepsilon \right) - \frac{RJa^2}{2\pi} \ln(0.387a) \tag{3.8}$$

if the pads are squares. In fact, these formulae appear in [5] with changed sign, because they compute the voltage drop, and not just the voltage at that point.

Although this formula is only an approximation, it gives good results when compared with other calculations. We have also checked this in our

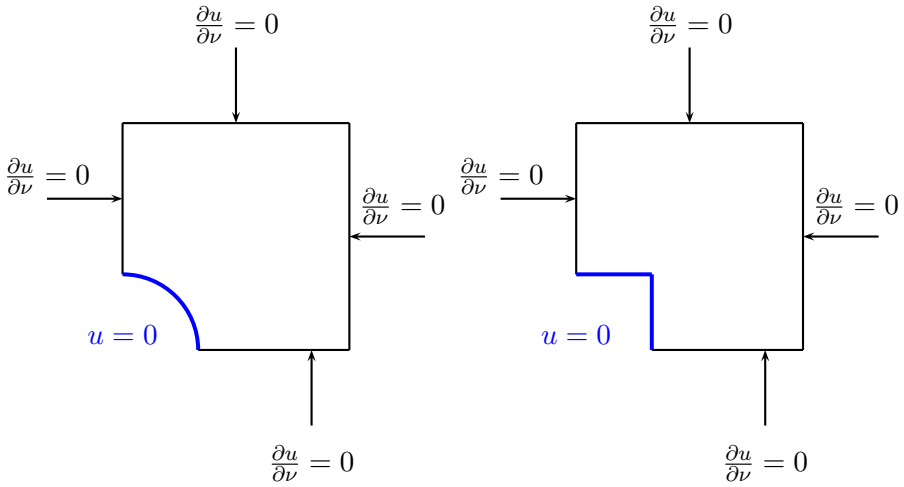


Figure 3.5: Boundary value problems.

finite element simulations, which we present in Section 3.5 below. The formulas are obtained in [5] with ad hoc arguments and reasonable estimates, but we want to point out that these formulas can be obtained systematically with the use of asymptotic analysis methods. We present this approach in the rest of this section.

If we write formula (3.7) when $a = 1$ and $RJ = 1$, we simply obtain

$$u(0.5, 0.5) \simeq \frac{1}{2\pi} \ln \varepsilon + 0.151. \tag{3.9}$$

One can imagine that this is just the beginning of an asymptotic formula in powers of ε like

$$u(0.5, 0.5) \simeq c_{-1} \ln \varepsilon + c_0 + c_1 \varepsilon + c_2 \varepsilon^2 + \cdots . \tag{3.10}$$

We show how the coefficients c_{-1} and c_0 of this formula can be obtained systematically, and will postpone the possibility of calculating the higher order values for future work. These coefficients will also be calculated using finite elements in Section 3.5 below.

In asymptotic analysis ([3] for example), one starts by distinguishing the *inner* and *outer* regions. The outer region will simply be the whole square $[0, 0.5] \times [0, 0.5]$ where we should solve the equation $\nabla^2 u^{\text{out}} = 1$ with the boundary condition $\partial u^{\text{out}} / \partial \mathbf{n} = 0$. This problem is clearly incompatible, and this is because by taking $\varepsilon = 0$ we have forgotten that the wall $\sqrt{x^2 + y^2} = \varepsilon$ acts like a sink that compensates the source term. So the correct outer

equation is $\nabla^2 u^{\text{out}} = 1 - \delta$ with the boundary condition $\partial u^{\text{out}}/\partial \mathbf{n} = 0$, where δ is the delta-function centered at $(0,0)$ and its coefficient has been chosen to make the system compatible.

It is perhaps easier to solve this problem by extending it to the full square $[0,1]\times[0,1]$ with one delta-function on each corner. Then the right-hand side of the equation will be the function $1-\delta_1-\delta_2-\delta_3-\delta_4$, that one can develop in double cosine series, and obtain

$$\sum_{\ell=1}^{\infty} 2\left(\cos(2\ell\pi x)+\cos(2\ell\pi y)\right) + \sum_{n,m=1}^{\infty} 4\cos(2n\pi x)\cos(2m\pi y).$$

So the solution is

$$u^{\text{out}} = C(\varepsilon) + u^{\text{aux}}(x,y),$$

where

$$u^{\text{aux}}(x,y) = \sum_{\ell=1}^{\infty} \frac{-1}{2\pi^2\ell^2}\left(\cos(2\ell\pi x)+\cos(2\ell\pi y)\right) + \\ + \sum_{n,m=1}^{\infty} \frac{-1}{\pi^2(n^2+m^2)}\cos(2n\pi x)\cos(2m\pi y)$$

and $C(\varepsilon)$ is a constant (depending on ε) that has to be determined.

Upon writing $r = \sqrt{x^2+y^2}$, we have evaluated these series numerically and calculated numerically the limit as $(x,y)\to 0$ of $u^{\text{aux}}(x,y) + (\ln r)/(2\pi)$ (remember that $\nabla^2 - \ln r = -2\pi\delta$) and have obtained the approximate value of -0.208. This means that

$$u^{\text{out}}(x,y) = -\frac{1}{2\pi}\ln r - 0.208 + O(r) + C(\varepsilon).$$

We have also calculated numerically $u^{\text{aux}}(0.5,0.5) \simeq -0.0552$. These numerical calculations have also been used to draw Figure 3.6.

On the other hand, the inner problem is now stated in the scaled variables $x' = x/\varepsilon$, $y' = y/\varepsilon$, and, letting $\varepsilon \to 0$, the region becomes

$$\Omega = \{(x')^2 + (y')^2 > 1,\ x > 0,\ y > 0\}.$$

Performing this change of variables in the equation and allowing $\varepsilon \to 0$, one finally has to solve

$$\begin{cases} (\nabla')^2 u^{\text{inn}} = 0 & \text{in } \Omega, \\ u^{\text{inn}} = 0 & \text{on } (x')^2 + (y')^2 = 1, \\ \dfrac{\partial u^{\text{inn}}}{\partial \mathbf{n}} = 0 & \text{on } x' = 0 \text{ and } y' = 0. \end{cases} \tag{3.11}$$

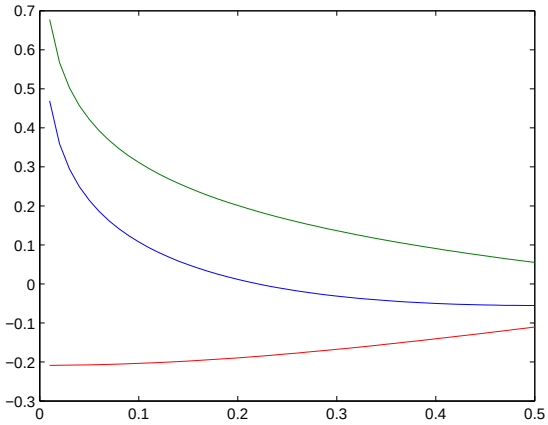


Figure 3.6: From top to bottom: $y = -\ln(\sqrt{2x^2})/(2\pi)$, $y = u^{\mathrm{aux}}(x,x)$, and $y = u^{\mathrm{aux}}(x,x) + \ln\sqrt{2x^2}/(2\pi)$.

The solution is

$$u^{\mathrm{inn}} = C_1 \ln r'.$$

Observe that C_1 does not depend on ε.

If we now write u^{out} also in the scaled variables, we see that one has to match $u^{\mathrm{out}} = -\frac{1}{2\pi}\ln(\varepsilon r') - 0.208 + C(\varepsilon) + O(\varepsilon)$ with $C_1 \ln r'$, and we obviously obtain that $C_1 = -1/(2\pi)$ and $C(\varepsilon) = \frac{1}{2\pi}\ln\varepsilon + 0.208 + O(\varepsilon)$. Finally,

$$u(0.5, 0.5) \simeq -0.0552 + C(\varepsilon) = \frac{1}{2\pi}\ln\varepsilon + 0.153, \tag{3.12}$$

which agrees with the formula (3.9) obtained in [5] up to the second significant digit.

It remains to be explained how we would proceed in the case of squared pads, represented in the left-hand picture of Figure 3.5. The outer solution is obtained as before and it is exactly the same solution. The difference lies in the outer solution, that has to be found in the domain

$$\Omega = \{x > 0,\ y > 0,\ \max(x, y) > 1\}.$$

This problem is more difficult than the previous one, but still solvable. With a Schwarz–Christoffel conformal map, one can transform the domain Ω into the half plane $y > 0$, and then, with (the inverse of) the Joukowsky map, the domain becomes the exterior of the unit disc, where the solution of (3.11) is simply $u^{\mathrm{inn}} = C_1 \ln r'$ again (see Figure 3.7 below).

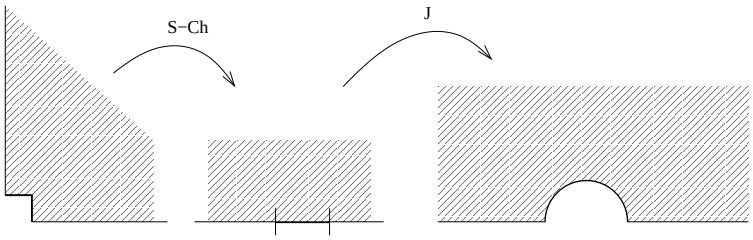


Figure 3.7: Two conformal transformations.

3.4 An averaged model

This section describes only projects that could be undertaken in the direction of building an averaged model. These projects are not very highly elaborated yet, and need further development.

From the mathematical point of view, the role of the pads in the equation of a flip-chip is to act as sinks to compensate the source terms of the main equation (3.1). Instead of solving boundary-value problems where the boundaries of all the individual pads would have to be taken into account, we could also look for an averaged homogenized or continuous model, where the role of the pads would be played by a new term in the equation (3.1), that would have to be solved in the whole chip, and not only outside the pads.

Moreover, we have so far considered only regular distributions of pads, but we could also consider less regular or even very disordered distributions, and there the use of the averaged equation would be even more useful. The same averaging principle could be applied to the sub-rectangles where the functions R and J are constant. If there are a large number of them, one can think that a kind of average would be relevant enough.

We are always thinking in approximating equations to help the designer reach an idea of how far his or her design is from a dangerous situation in which the specifications may not be fulfilled. Of course, once the chip is fully designed, it is reasonable and necessary to perform a full simulation involving all the pads and all the sub-rectangles, with all the details. But this type of simulation is perhaps not very reasonable for carrying it out *online* in the design process.

In order to simplify the problem when there are many pads and many rectangular regions where the function $J(x, y)$ is constant, it is a good idea to forget the existence of the ground pads and the zero boundary condition

on them, and try to represent their effects through a new dissipation term $-Ku$ in the equation, that could become

$$\nabla \cdot \left(\frac{1}{R(x,y)} \nabla u \right) - Ku = J(x,y), \tag{3.13}$$

where K can either be a constant or a space-dependent coefficient.

This kind of process of averaging has quite a large tradition in the mathematical literature, under the name of *homogenization in PDE's*. For a recent publication with some aspects of this approach one can see the monograph [4].

In that approach, a bulk material represented by a set Ω and some *absorbing inclusions* $F_\varepsilon \subset \Omega$ is considered. This is the same as our case, which is different from a situation where there are Neumann boundary conditions at the interior inclusions, the so-called *reflecting inclusions*. Nevertheless, it has to be remarked that when $\varepsilon \to 0$ it is supposed that not only the diameters of the inclusion components tend to zero, but also the distances among these components tend to zero in the correct proportion. In the notation suggested by Figure 3.4, one would have something along the lines of $a = h\varepsilon$, for a fixed h, and then $\varepsilon \to 0$.

Another possibility would be to change the right-hand side of (3.1) instead of the left-hand side. This has more or less been the way to deal with the outer solution in the preceding section. In the calculation of the outer solution we have substituted the effect of the (small) pads by the effect of delta functions. That could be another option to explore. It should also be noted that with these outer solution we have not calculated more than the first two terms in the series expression (3.10). The way in which the subsequent terms are calculated would also give more insight in the development of a more refined averaged model.

3.5 Simulations with finite elements

The type of problems that we have dealt with so far are easy to tackle using some standard numerical methods such as Finite Element Methods (FEM). Indeed, one would start by computing some solutions to both the flip-chip package problem or the wire-bond one to get some insight on the type of solutions to be expected and even to compute an excellent numerical approximation of the maximum IR-drop in the chip. However, as it was already exposed in the Introduction, the idea is to provide the designer with formulae such that with a simple scientific calculator he or she could obtain an approximation to the maximum IR-drop with no need of solving any differential equation by any means. In this sense, the sort of expressions given

in (3.9)–(3.10) are the type of formulae that we are looking for. There are analytic asymptotic method techniques that give the right values of the a-priori unknown coefficients $c_{-1}, c_0, c_1, \ldots$ Nonetheless, and given that these equations are easy to solve numerically, another possibility is to numerically obtain the maximum IR-drop and afterwards match the obtained value with the formula given in (3.10) to obtain the coefficients.

In this section, we are going to use the FEM method to solve the equations and we are going to show how to obtain the coefficients in (3.10) afterwards. Let us focus on the elementary problems stated in Figure 3.5. In particular, let us consider the following problem:

$$\begin{cases} \Delta u = 1 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega_1, \\ u = 0 & \text{on } \partial\Omega_2. \end{cases} \tag{3.14}$$

Here we are taking $R(x,y)\,J(x,y) \equiv 1$ in the whole domain. We are considering $\Omega = \Omega_Q \setminus \Omega_\varepsilon$, where $\Omega_Q = (0,\, 0.5) \times (0,\, 0.5)$ and Ω_ε stands for the pad domain. We distinguish two cases here: $\Omega_\varepsilon = \{\max(x,y) \leq \varepsilon\}$ in the case of a square pad of length ε, and $\Omega_\varepsilon = \{x^2 + y^2 \leq \varepsilon^2\}$ in the case of a circular pad of radius ε. The values of ε that we have considered are $\varepsilon = 0.05, 0.1, 0.15, 0.20$. Concerning the boundaries, $\partial\Omega_2$ stands for $\partial\Omega_\varepsilon$ and $\partial\Omega_1 = \partial\Omega \setminus \partial\Omega_2$.

This type of problems can be solved numerically using, for instance, the `pdetool` toolbox from `Matlab`. This is a FEM software which has some standard PDE's implemented that can be numerically solved for different geometries in $\mathbb{R}^2$. We need to follow these steps:

1. Define the geometry of the problem.
2. Define the boundary conditions, that can be different depending on the part of the boundary.
3. Choose the PDE between the ones that are already implemented and give the values of the parameters involved.
4. Initialize the mesh and refine it if necessary.
5. Solve the PDE problem.

In Figure 3.8 we present some screenshots of these previous steps in `Matlab`. Once we have done this, we obtain the numerical solution for the PDE on each node of the mesh. The software also allows us to plot this solution and its contour levels (see Figure 3.9).

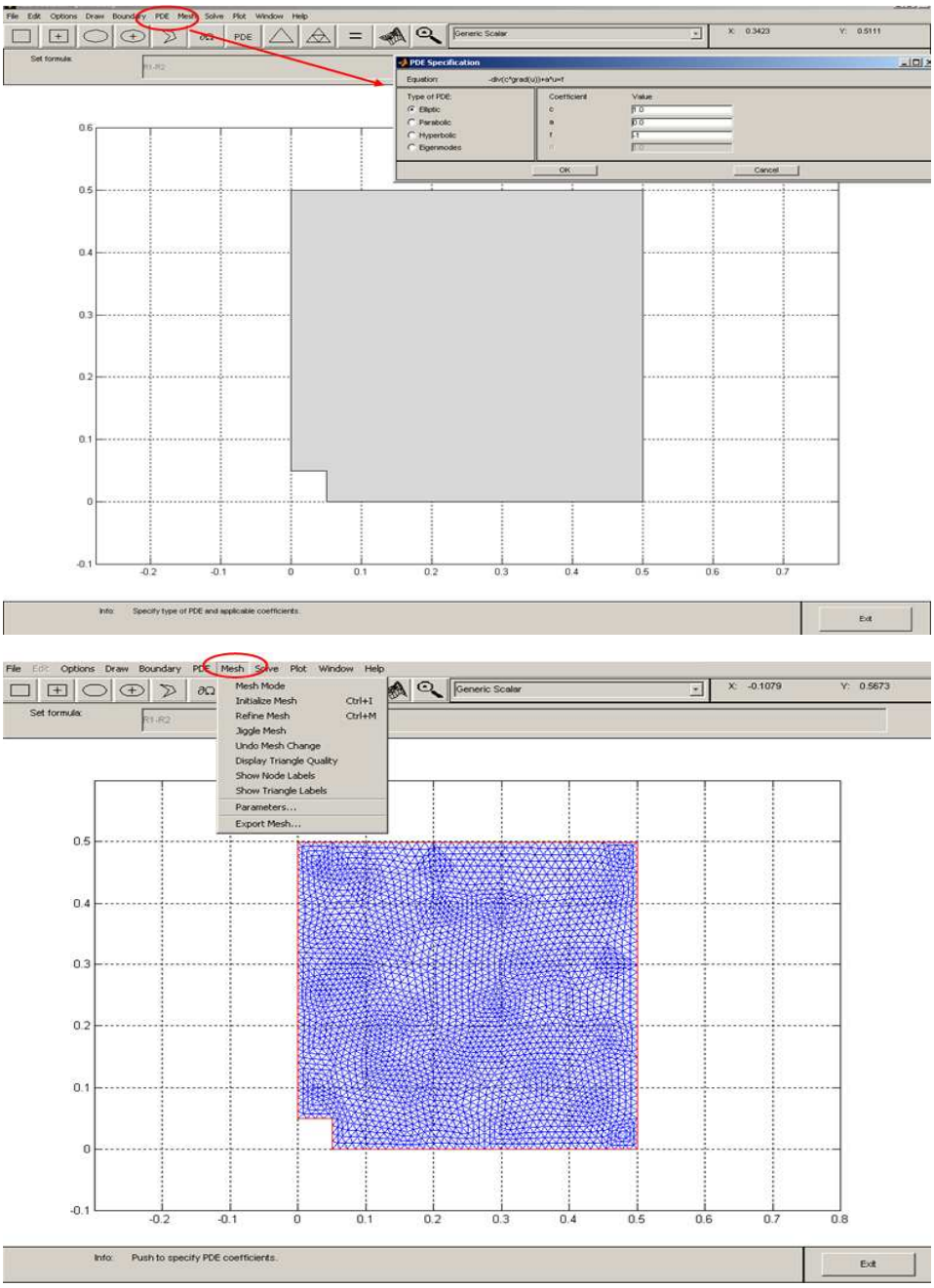


Figure 3.8: Two screenshots showing the aspect of the simulation with `Matlab`. From top to bottom, PDE and mesh definition for a problem with a square pad.

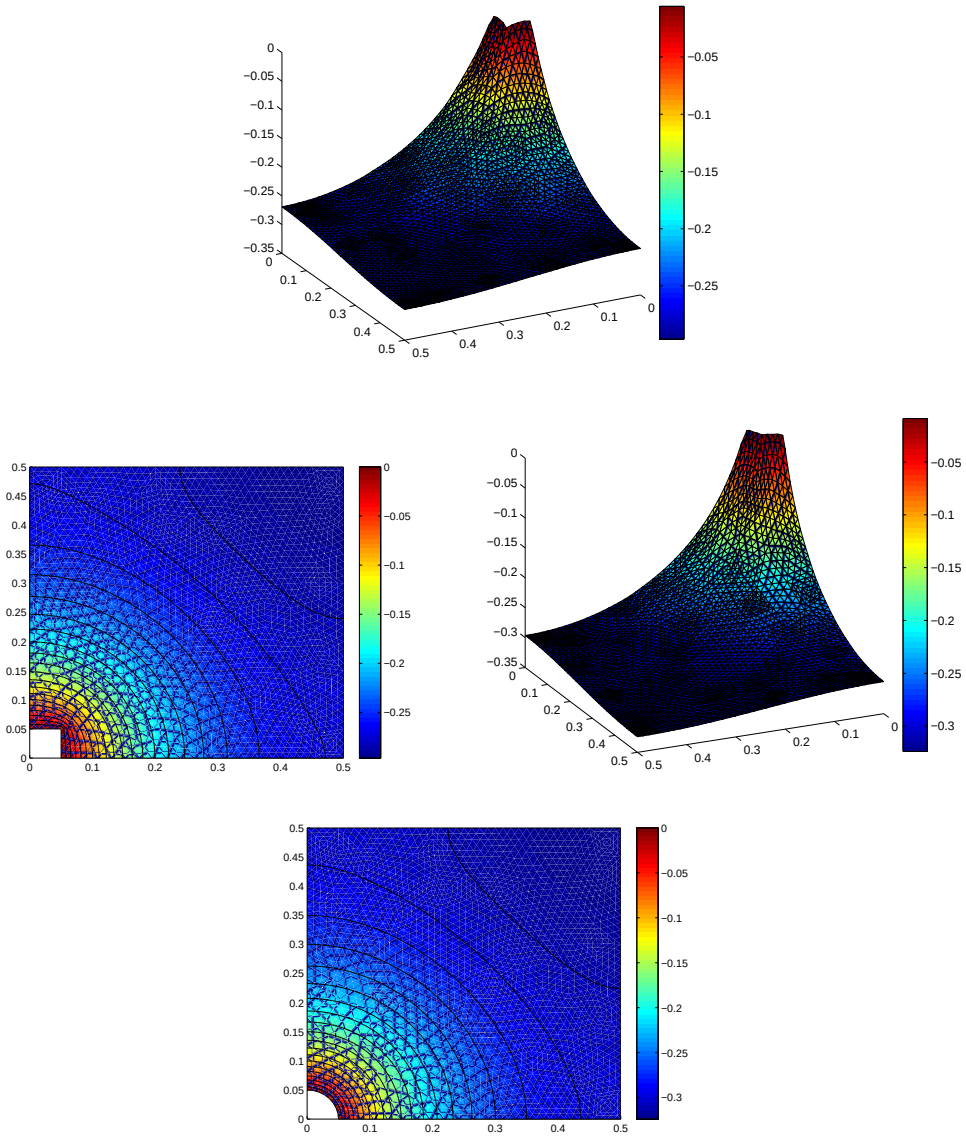


Figure 3.9: Numerical solutions, including contour plot, for a square and a circular pad when $\varepsilon = 0.05$.

It is also interesting to note that, once achieved these numerical values at the nodes of the mesh, one can manipulate them in order, for instance, to calculate the maximum of the absolute value of the solution, which in this case is attained at $u(0.5, 0.5)$ (see Section 3.3). This, together with the asymptotic expansion obtained in (3.10), may be used to compute approximate values for the coefficients c_0, c_1 and c_2 (according to Section 3.3, we will take $c_{-1} = 1/2\pi$, as $R, J, a = 1$ in this problem). To do so, one has to use the numerically obtained values of $u(0.5, 0.5)$ for the different values of ε and find the set of coefficients in (3.10) that best fit them. For instance, in this case, the following system must be solved:

$$\begin{cases} -0.323992 = \frac{1}{2\pi}\ln 0.05 + c_0 + c_1(0.05) + c_2(0.05)^2 \\ -0.215547 = \frac{1}{2\pi}\ln 0.1 + c_0 + c_1(0.1) + c_2(0.1)^2 \\ -0.154141 = \frac{1}{2\pi}\ln 0.15 + c_0 + c_1(0.15) + c_2(0.15)^2 \\ -0.112725 = \frac{1}{2\pi}\ln 0.2 + c_0 + c_1(0.2) + c_2(0.2)^2 \end{cases} \tag{3.15}$$

for a circular pad, and

$$\begin{cases} -0.297656 = \frac{1}{2\pi}\ln 0.05 + c_0 + c_1(0.05) + c_2(0.05)^2 \\ -0.190042 = \frac{1}{2\pi}\ln 0.1 + c_0 + c_1(0.1) + c_2(0.1)^2 \\ -0.129616 = \frac{1}{2\pi}\ln 0.15 + c_0 + c_1(0.15) + c_2(0.15)^2 \\ -0.089079 = \frac{1}{2\pi}\ln 0.2 + c_0 + c_1(0.2) + c_2(0.2)^2 \end{cases} \tag{3.16}$$

for a square pad. As these systems are overdetermined, the idea is to find the coefficients c_0, c_1 and c_2 that are *closer* to the exact solution upon using a linear least squares method (see for instance [2]). Hence, one has to solve the corresponding normal equations, which in this case yield

$$c_0 = 0.153420,\ c_1 = -0.00005163,\ c_2 = -0.249654, \tag{3.17}$$

for a circular pad, and

$$c_0 = 0.180623,\ c_1 = -0.0168976,\ c_2 = -0.254666, \tag{3.18}$$

for a square pad.

These results are in strong agreement with those of [5] included in Section 3.3. For circular pads, formula (3.7) is written for $a = 1$ and $RJ = 1$

in (3.9). For square pads with $a = 1$ and $RJ = 1$, formula (3.8) becomes

$$u(0.5, 0.5) \simeq \frac{1}{2\pi} \ln \varepsilon + 0.178.$$

However, the good point in our calculations in comparison to the method used in [5] is that the method we have used would hold to obtain higher order terms in the expansion, and thus obtain a more precise solution. This is especially interesting since, in real chips, the pads are not that small in comparison with the size of the chip, and hence the value of ε is actually not so small. Thus, the only chance to obtain greater accuracy is to take higher order terms in the asymptotic expansion presented in Section 3.3.

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