Limit behavior in a wave equation for a spring-mass-damper system

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MATHEMATICS 2005

4-7 April, Liverpool

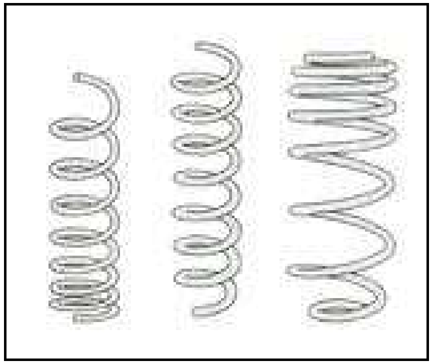
Outline

- The spring-mass-damper **system**.
- The **model**: wave equation with strong damping.
- Main **tools**: dominant eigenvalues, generalized convergence.
- Main **results** ...

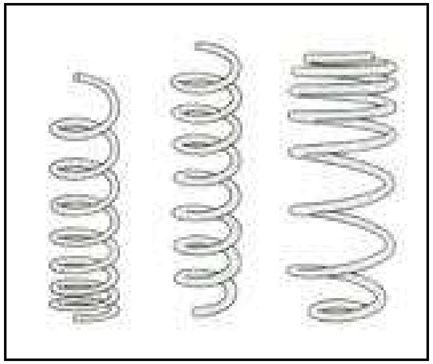
The system

A damped spring-mass system

Present in: automotion, seismic control, ...



Spring or **viscoelastic** device
(elasticity + possible **internal** dissipation)
fixed at one end



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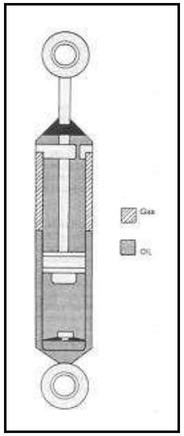
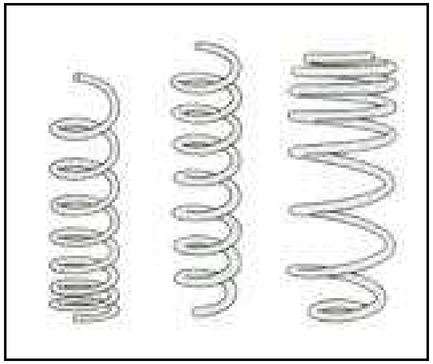


Damper or **viscous** device
(**external** dissipation)
acting onto the mass at the other end



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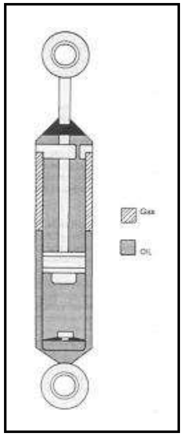
⇒ Equation



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⇒ Boundary conditions

Motivating the problem: PDE vs. ODE

Spring-mass system as an ODE (spring as a discrete system):

$$m\,u''(t) + r\,u'(t) + k\,u(t) = 0$$

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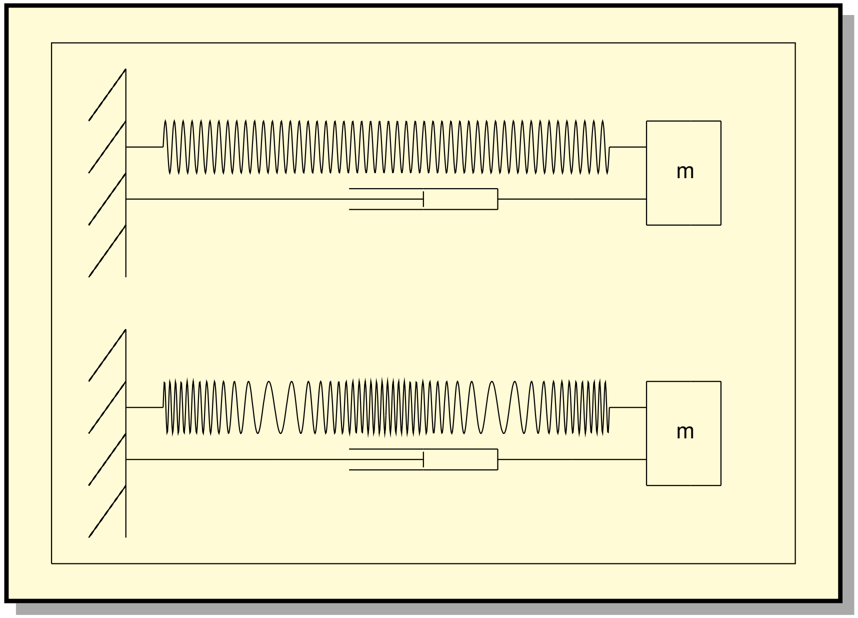
But ...



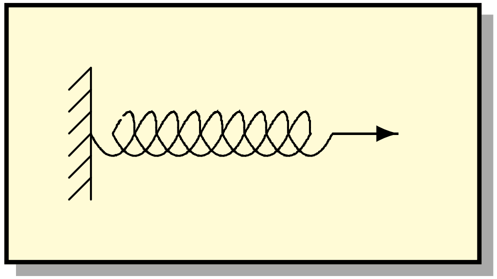
- differences in internal deformation?
- possible internal dissipation?

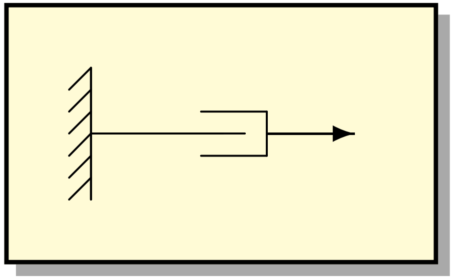
Modelling

- Momentum balance law
- **Rheological** approach

spring (elasticity) and **dashpot** (viscosity)
combined in series or parallel

$$\boxed{\sigma_e = E\,\varepsilon_e} \qquad \boxed{\sigma_v = E_1\,\dot{\varepsilon}_v}$$





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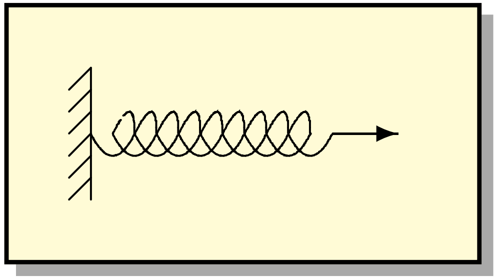
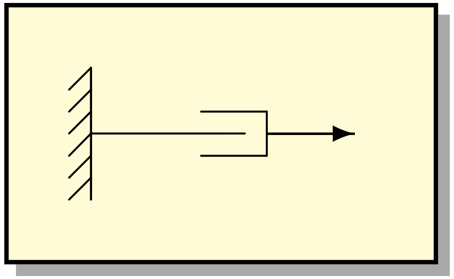
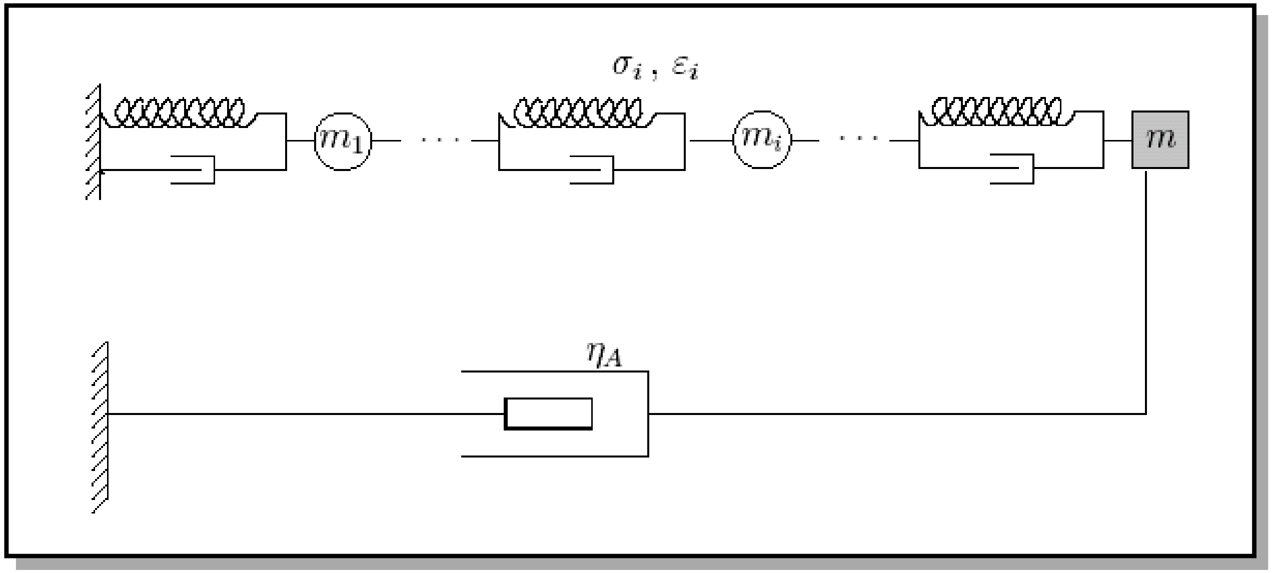




Continuous Kelvin-Voigt model
or
wave equation with strong damping
(our case)



Rheological model:

E, E_1 *spring elasticity and viscosity;* q *external viscosity.*

Linear (and dimensionless) model

Wave eq. with **strong damping** and
dynamical boundary conditions:

$u(x,t)$ = displacement of the x particle at time t

$$\begin{cases} u_{tt} - u_{xx} - \alpha\, u_{txx} = 0, \quad 0 < x < 1, \ t > 0 \\ u(0,t) = 0 \\ u_{tt}(1,t) = -\varepsilon \ \left[u_x(1,t) + \alpha\, u_{tx}(1,t) + r\, u_t(1,t)\right] \end{cases}$$

$\alpha \geq 0 \rightsquigarrow$ internal damping (spring)[a]

$r > 0 \rightsquigarrow$ external dissipation (external damper)

$\varepsilon \geq 0 \rightsquigarrow$ inverse of the external mass[b]

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[a] ***"Internal viscosity and limit behavior in a spring-mass model"***, MP, J. Sola-Morales. Preprint (2005)

[b] ***"Analysis of a viscoelastic spring-mass model"***, MP, J. Sola-Morales. JMAA (2004)

[c] ***"Analisi d'un model de suspensio-amortiment"***, MP, (Thesis UPC, 2004)

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- ***M. Grobbelaar-van Dalsen (1994):*** $\alpha = \varepsilon = 1$ ***,*** $r = 0$ *; P. Massat (1983).*
- ***D.L. Russell (1986):*** *explicit case* $\varepsilon = 0$ *for elastic beam equation.*

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- ***Elastic medium + rigid mass:*** *C.Castro, E.Zuazua '98 (controllability) ; ...*
- ***Wave eq. with strong damping:***
  N. Cónsul, J. Solà-Morales '99 (equilibria stability) ;
  K. Liu, Z.Liu '98; S. Chen, K. Liu, Z.Liu '99 (exponential decay).
- ***Wave eq. with weak damping + dynamical b.c.:***
  A. Freiria Neves, H. de Souza Ribeiro, O. Lopes '86.
- *...*

Linear (and dimensionless) model

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- ***Non-Homogeneous version*** [a]*: globally bounded solutions ⇝ Ressonances*
- ***Non-Linear problem*** [b]*: feedback control, exponentially attracting invariant manifold (A.N. Carvalho, 1995; A.N. Carvalho, G. Lozada-Cruz, 2001)*

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[a] *Asymptotic behavior and oscillations in a viscoelastic problem*, MP, Proceedings XVIII CEDYA-VIII CMA (2003).

[b] *In preparation*, MP.

Main tools

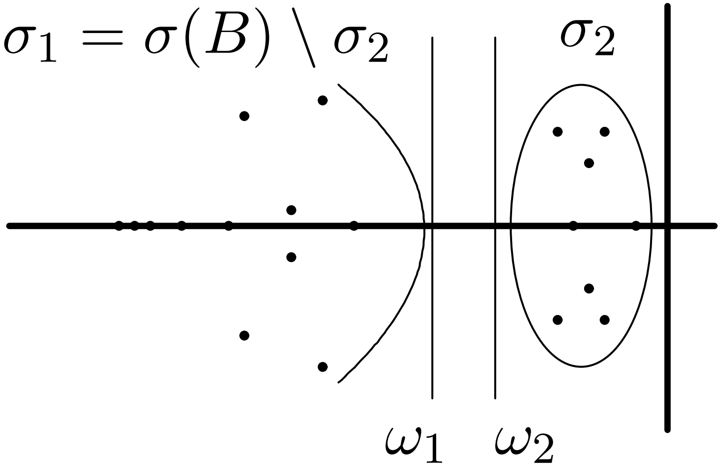
1. Asymptotic behavior of solutions: reduction to ODEs.

Take the evolution equation $\boxed{x'(t) = B\,x(t)}$ with B analytic.

Def: We call σ_2 a set of **dominant eigenvalues of** B if:

$\sigma(B) = \sigma_1 \cup \sigma_2$, where σ_2:

- finite subset of isolated eigenvalues
- finite algebraic multiplicity
- $Re\,(\sigma_1) < \omega_1 < \omega_2 < Re\,(\sigma_2)$ (if $Re\,(\sigma_2) = C$, we call σ_2 the **maximal dominant** set).



Thm: In this case, the equation of finite dimension $\boxed{x_2' = B_2\,x_2}$ can be thought as the **limit** when $t \to +\infty$ of $x' = B\,x$

2. **Generalized convergence of operators** (T. Kato).

- Generalization of norm-convergence ...

- **Thm [T. Kato] (Semicontinuity of separated parts of the spectrum):**

  Let T closed and let $\sigma(T)$ be **separated** *in two parts by a closed curve Γ. For every closed operator S s.t. $\hat{\delta}(T, S)$ is small enough, we have $\sigma(S)$* ***separated in the same way***

  (isomorphic eigenspaces, norm convergence of projections, ...)

$$\Downarrow$$

continuity of finite sets of eigenvalues ...

Main results

We can write the PDE model as an evolution equation:

$$\boxed{\frac{d}{dt}V - A_\alpha(\varepsilon)V = 0}$$

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PDE → classical ODE ? ⇔ dominant eigenvalues?

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$\boxed{\text{PDE} \longrightarrow \text{classical ODE ?}}$ $\Leftrightarrow$ $\boxed{\text{dominant eigenvalues?}}$

Two points of view (**physical** and **mathematical** interest):

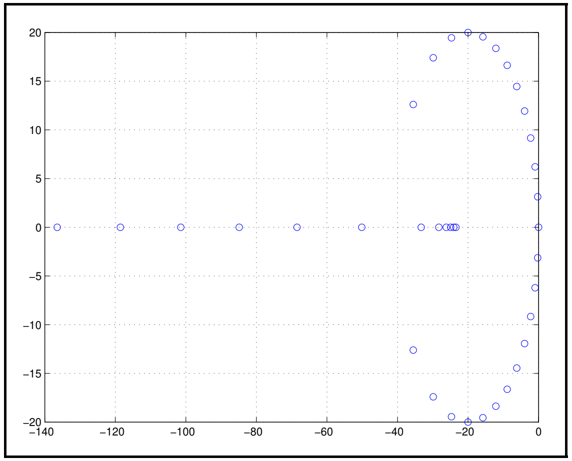
- ε (inverse of the mass): $\varepsilon \sim 0$.
- α (internal viscosity):
  - $\alpha = 0$
  - $\alpha \sim 0$
  - $\alpha >> 1$

1a. Infinite mass at the end or $\varepsilon = 0$ ($\alpha, r > 0$).

Explicit solutions

($\lambda(0) = 0$ double **dominant** eigenvalue)

and $\sigma_{ess} = \frac{-1}{\alpha}$



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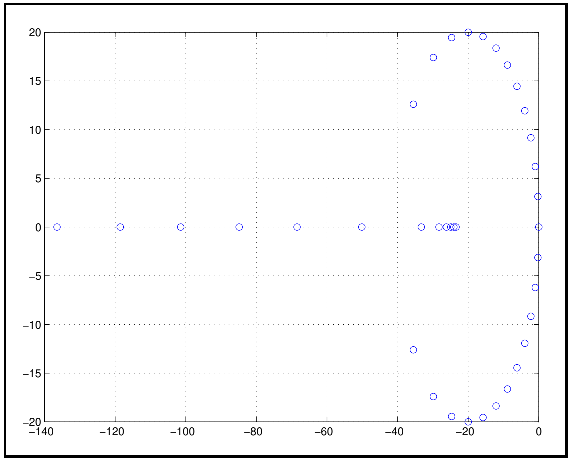
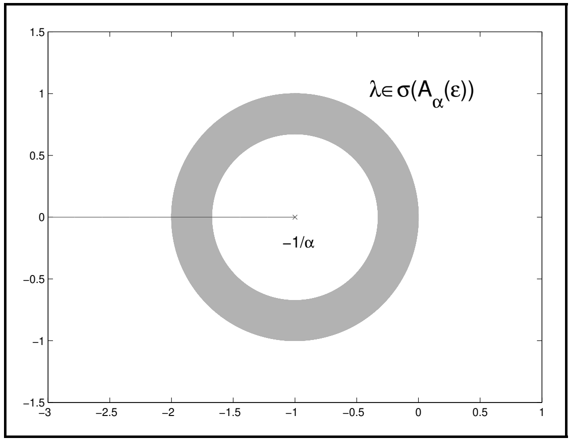


1b. Large mass at the end or small $\varepsilon > 0$ ($\alpha, r > 0$).

- **Thm:** Spectrum as a perturbation of the limit case (when $\varepsilon\alpha r < 1$).
- **Thm:** Existence of **2 dominant eigenvalues** $\{\lambda_0^+(\varepsilon), \lambda_0^-(\varepsilon)\}$ if ε small enough

  (explicit perturb. of $\lambda(0) = 0$)



Thm (limit ODE):

For m sufficiently large, the **asymptotic behavior** of the PDE solutions at $x = L$ is given by the solutions of the following 2nd order ODE:

$$\boxed{m\, w'' + k_1\, w' + k_0\, w = 0 \quad \textit{(dimensional)}}$$

where $w(t) = u(L,t)$ and

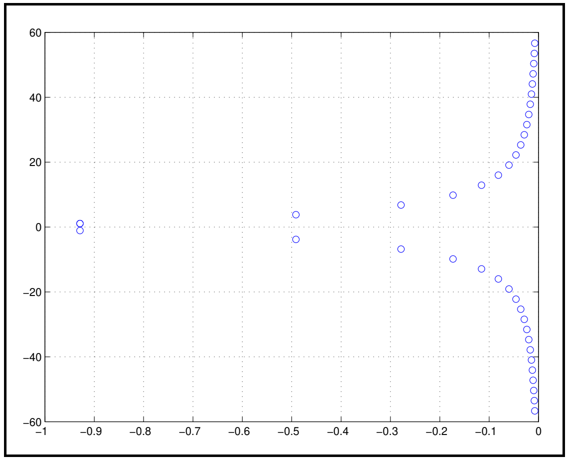
$$k_1 = \left(\frac{E_1}{L} + q\right) - \frac{1}{3}\left(\frac{E_1}{L} + q\right)\left(\frac{\rho L}{m}\right) + \left(\frac{4\,E_1}{45\,L} + \frac{q}{15}\right)\left(\frac{\rho L}{m}\right)^2 + \dots$$

$$k_0 = \frac{E}{L}\left[1 - \frac{1}{3}\left(\frac{\rho L}{m}\right) + \frac{4}{45}\left(\frac{\rho L}{m}\right)^2 + \left(\frac{q^2}{45\,E\rho} - \frac{16}{945}\right)\left(\frac{\rho L}{m}\right)^3 + \dots\right]$$

PDE $\longrightarrow$ ODE, but observe the coefficients !!

2a. Purely elastic spring or $\alpha = 0$ ($\varepsilon, r > 0$)

- A_0 generator of a $\mathcal{C}^0$ semigroup of contractions.
- Spectrum:
  - $\sigma(A_0) = \sigma_p(A_0)$ (not essential spectrum)
  - infinitely many, isolated, finite algebraic multiplicity.
  - $Re\,\lambda < 0$ for all $\lambda \in \sigma(A_0)$.
  - $Re\,\lambda_n \to 0$ and $Im\,|\lambda_n| \to \infty$.



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*(**Proof:** $-A_0$ maximal monotone operator)*

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  - $Re\,\lambda_n \to 0$ and $Im\,|\lambda_n| \to \infty$.

*(**Proof:** Analysis of eigenfunctions,
characteristic equation for eigenvalues and energy)*

Thm (Asymptotic behavior of solutions):

- Not existence of any finite subset of dominant eigenvalues:

  $$\boxed{\text{PDE} \nrightarrow \text{ODE!!}}$$

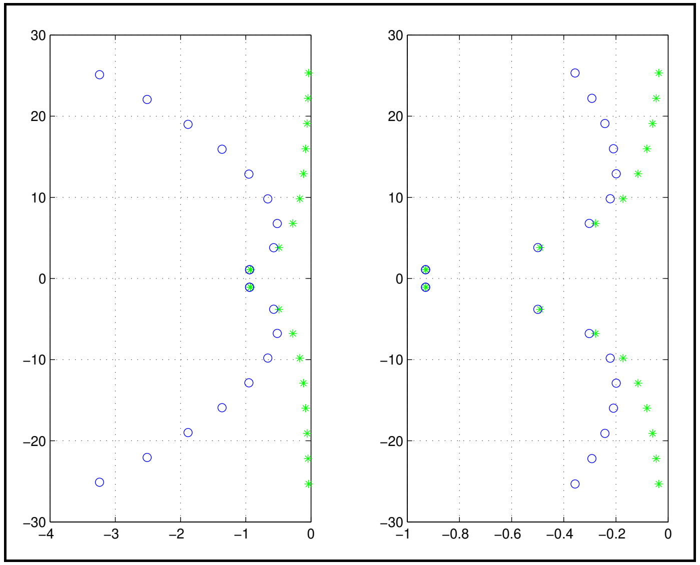
- But ... all solutions **tend to 0** when $t \to +\infty$.

  And there exist solutions which tend to zero **as slow as we wish** (because no internal damping).

  Polinomial decay for smoother solutions?

2b. Viscoelastic spring with small $\alpha > 0$ ($\varepsilon, r > 0$)

- A_α is the generator of an analytic semigroup if $\alpha > 0$ (and $\varepsilon \geq 0$).
- Spectrum:
  - $\sigma_{ess}(A_\alpha) = \{-1/\alpha\}$, $\alpha > 0$;
  - $\sigma_p(A_\alpha) \underset{\alpha \to 0}{\longrightarrow} \sigma_p(A_0)$ in compact sets, but **not globally**.



$(\alpha = 0.01)$ $(\alpha = 0.001)$

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*(**Proof:** Massat'83)*

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*(**Proof:** $A_\alpha \underset{\alpha \to 0}{\longrightarrow} A_0$ in a generalized sense,*
*but A_α is **sectorial** when $\alpha > 0$)*

Thm (Existence of dominant eigenvalues and limit ODE):

If $\alpha > 0$ is small enough,

there **exists** a finite subset of dominant eigenvalues, but ...

$\Downarrow$

$\boxed{\text{PDE} \rightarrow \text{ODE ?}}$

$\Downarrow$

Yes, but the **order** and **coefficients** of the ODE
may be differents for each α.

(the limit ODE may **not** be of order 2 !!)

2c. Overdamped viscoelastic spring or $\alpha > 0$ large

- Explicit limit case when $\varepsilon = \infty$, $r = 0$: **overdamping** for α large.

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- **Thm (No limit ODE):**

  If ε large, r small and α **large enough**:

$$-\infty < \lambda_n < -1/\alpha \quad \text{with} \quad \lambda_n^+ \to -1/\alpha$$

$$\Downarrow$$

  - No finite subset of dominant eigenvalues $\Rightarrow$ $\boxed{\text{PDE} \not\to \text{ODE!!}}$
  - **Overdamping** phenomena

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  - **Overdamping** phenomena

*(**Proof:** $z = \sqrt{1 + \lambda\alpha}$ in char. eq. + comparison with $e^{2\frac{z^2-1}{\alpha z}} = -1$)*