Intensive Programme	**Mathematical Models in Life and Social Sciences**
Course:	**Structured Population Dynamics in ecology and epidemiology**
Lecturer:	Jordi Ripoll (Universitat de Girona, Spain)
Dates and place:	*September 7-19 2009 – Alba Adriatica, Italy*

Abstract:

Structured population dynamics deals with the time evolution of the population composition with respect to the state of the individuals. Typically, the structuring variables are age (with different interpretations, e.g. chronological or time since infection or time since cell division), size, phenotype, maturity level, space, etc. The simplest structure is given by the age of the individuals since the evolution of age over time proceeds with speed one. Moreover, age-structured population models play a central role since, in some cases, models considering other physiological characteristics can be reduced to them. The course encompasses (describes/analyzes) different types of models: unstructured and structured, continuous and discrete, all of them coming from ecology (highlighting the interactions between individuals and their environment) and epidemiology (e.g. the spread of infectious diseases). The course is divided in 4 parts: an introduction to the abstract setting for the linear case (semigroup approach); early ecological and epidemiological models (ordinary differential equations); continuously structured population models (partial differential equations/integral equations); and matrix population models (discrete structure). Background: basic knowledge of ordinary differential equations and linear algebra.

Program:

1. Introduction: time-continuous linear population dynamics. Differential equation and Integral equation. Semigroup approach.

2. Early ecological models. Exponential/logistic population growth. *Allee* effect. Probabilistic interpretation.

3. Early epidemiological models. Structure according to disease stage: SIS and SIR models. Epidemic threshold theorem. Basic reproductive number R_0.

4. Continuously age-structured population models. Lotka-Mckendrick equation. Integration along characteristics. Renewal equation. Asynchronous exponential growth. “Stable” age distribution. (Generalized) Gurtin-MacCamy equation. Equilibria and linearization. Kermack-Mckendrick equation.

5. Size-dependent problems. Change of variables. Reduction via linear chain trick.

6. Numerical Simulations. Discretization. Periodic orbits.

7. Matrix population models. Leslie/Usher/*Tridiagonal* models. Perron-Frobenious theory. Fundamental theorem of demography. Basic reproduction number R_0. Fibonacci sequence.

Bibliography: books

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Material not covered in the course: delay-differential equations, Lotka-Volterra type models, Biological motion and chemotaxis, Adaptive dynamics of phenotypic traits, Selection-Mutation equations, and Complex networks.

Final Test (Master Students):

Consider the following linear problem of the Fibonacci's rabbits in continuous age and time:

$$u_t(a,t) + u_a(a,t) = 0\,, \quad u(0,t) = \int_2^\infty u(a,t)\,da\ , \tag{1}$$

where $u(a,t)$ is the age-density of pairs of rabbits. Notice that there is no mortality term and the birth rate (the influx of newborns) is given by the mature pairs (age $a \geq 2$).

1. Find the eigenvalues and the eigenfunctions of the linear system (1). Hint: compute the solutions with separate variables.
2. Compute the "stable" age distribution of the pairs of rabbits and $\lim_{t\to\infty} \frac{P(t+1)}{P(t)}$, where $P(t)$ is the total population.

Exercises of the slides:

page 6 For the Allee model, find the relation between population x and time t (implicit solution).

page 8 Draw the qualitative behaviour of the time-continuous models (Malthus, Verhulst, Allee).

page 11–12 Find the relation of the parameter β with respect to infectiveness, total population and the contact rate when transmission is limited (e.g. $c = 1$) and when it is non-limited (e.g. $c = N$). Find the units of all parameters in both cases.

page 14 Compute and interpret the growth rate r and the carrying capacity K for the particular logistic equation of the SIS model.

page 17–18 Integrate Lotka-Mckendrick system (1) along the characteristic lines $a = t + c$ and obtain system (2). Hint: write a linear ode for $\nu(t) := u(t+c,t)$ and solve it with initial condition at $t_c = \max\{0, -c\}$.

page 18 Obtain the solutions with separate variables letting $u(a,t) = T(t)\,A(a)$. Show that any complex solution $\lambda = x + yi$ of the characteristic equation is such that $\mathrm{Re}(\lambda) < \alpha^*$. Hint: use the properties of the complex exponential function.

page 19 Derive the renewal equation (3) from the boundary condition of (1) and using system (2).

page 21 Solve the "linear" ode for the normalized eigenfunction of the adjoint operator.

page 26 Write an extension of the SIS model with infection age as the structuring variable.

page 28 Find the expressions for the new fertility and the new mortality for the reduced age-dependent problem.

page 29 Check the terms appearing in the matrix of the ode. Hint: multiply the equation by a suitable weight function in age and then integrate over the age span.

page 30 Write the numerical scheme for the case of the Lotka-Mckendrick (linear) equation. Is it an implicit scheme?

page 33 Write the time-discrete version of the SIR and SIS models [pages 12 and 14]. Hint: discretize the ode and convert rates into probabilities. You can obtain a linear system if the density-dependent probabilities are fixed by the initial condition.

page 36 Show the Fundamental theorem of demography when the projection matrix is such that $P = W \cdot \Lambda \cdot V^T$.

page 39 Check the expressions of R_0 for the Leslie and Usher matrices. Hint: compute the matrix R.

page 41 Write the linear recurrence equation for the total number of pairs of rabbits as a two-dimensional matrix population model.