Statistical modelling on coordinates

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Note

This paper is a first draft of the principle of statistical modelling on coordinates. Several causes—which would be long to detail—have led to this situation close to the deadline for submitting papers to CODAWORK'03. The main of them is the fast development of the approach along the last months, which let appear previous drafts as obsolete. The present paper contains the essential parts of the state of the art of this approach from my point of view. I would like to acknowledge many clarifying discussions with the group of people working in this field in Girona, Barcelona, Carrick Castle, Firenze, Berlin, Göttingen, and Freiberg. They have given a lot of suggestions and ideas. Nevertheless, there might be still errors or unclear aspects which are exclusively my fault. I hope this contribution serves as a basis for further discussions and new developments.

1 Introduction and motivation

Historically, real statistics has been developed for real random variables or vectors, *i.e.* for functions going from a probability space to real space $\mathbb{R}^p$, $p \geq 1$. If we look *e.g.* in *Grundbegriffe der Wahrscheinlichkeitsrechnung* (Kolmogorov, 1946), published originally in 1933, we find both the real line, $\mathbb{R}$, and $\mathbb{R}^p$ as examples of infinite, continuous probability fields. $\mathbb{R}^p$, $p \geq 1$, is understood as the set of vectors of real numbers, whose elements follow rules which are summed up in the definition of Euclidean space. Consequently, we are used to add vectors, to multiply them component by component, to multiply them by a constant, to compute the scalar product of two vectors, to determine the length or norm of a vector, or to compute the distance between elements of the associate affine space, without even thinking if this is the proper way to handle our data. Furthermore, we have powerful tools, like integration and derivation, functional relationships, linear algebra, and many more, which allow to solve many problems, at least in an approximate way, although sometimes in a rather complicated manner.

But a closer look at the multidimensional example given by Kolmogorov on p. 18, shows that the author refers to $\mathbb{R}^p$ as the p-dimensional Euclidean space of *coordinates*. And the general theory of linear algebra tells us that any real Hilbert space has an orthonormal basis with respect to which the coefficients or coordinates behave like usual elements in real space, satisfying all the rules mentioned. Hence the usage of the term *Euclidean space* for all those spaces in the finite dimensional case, identifying the properties of $\mathbb{R}^p$ with the properties of the original space. It implies that properties that hold in the space of coordinates transfer directly to the original space. Of particular interest to us is the fact that concepts like Borel sets, probability and Lebesgue measure, Borel measurable function, probability distribution and density function—to mention just a few—can be taken as defined on the coordinates or coefficients with respect to an orthonormal basis. In other terms, reinterpreting Kolmogorov's words in an algebraic sense, *statistical analysis in an arbitrary Euclidean space can be identified with conventional statistical analysis on the coefficients with respect to an orthonormal basis.*

One might argue that this fact is of little interest in everyday practice, as usual observations *are* real numbers—*i.e.*, they are registered as coefficients of the canonical basis of $\mathbb{R}^p$—and hence *have* to be analyzed using the rules developed for the Euclidean space $\mathbb{R}^p$. We do not deny the fact that usual observations *are* in fact real numbers—*i.e.*, that they are registered as coefficients of a canonical basis of $\mathbb{R}^p$—, but we suggest that they *do not have* to be necessarily analyzed applying the rules developed for the Euclidean space $\mathbb{R}^p$ to the observations themselves. This reasoning is based on the fact that many subsets of real space, like *e.g.* the positive real line, the positive octant of the real plane, the $(0, 1)$ interval, the unit square, the sample space of compositional data—data whose measurement units are parts of some whole, like parts per unit, percentages or ppm—, or their generalizations, can be structured as Euclidean spaces. Consequently, in these very common cases it is possible to apply standard statistical theory to the coefficients in an orthonormal basis. Results differ from standard theory, but many properties are surprisingly easy to obtain. The main problem is interpretation, which can be attempted through formulation in terms of the canonical basis of $\mathbb{R}^p$ of usual properties given in terms of coefficients with respect to an orthonormal basis. First theoretical steps and some examples can be found in Pawlowsky-Glahn and Egozcue, 2001, 2002; Mateu-Figueras, Pawlowsky-Glahn, and Martín-Fernández, 2002; Egozcue, Pawlowsky-Glahn, Mateu-Figueras, and Barceló-Vidal, 2003; Mateu-Figueras and Pawlowsky-Glahn, 2003; von Eynatten, Barceló-Vidal, and Pawlowsky-Glahn, 2003; von Eynatten, Barceló-Vidal, and Pawlowsky-Glahn, 2003; and Pawlowsky-Glahn, Egozcue, and Burger, 2003. For an example in spatial statistics see Tolosana-Delgado and Pawlowsky-Glahn, 2003, 2003. An extensive development for the compositional case is to be found in Mateu-Figueras, 2003. In this contribution we present a synthetic approach to the theoretical background and some properties on particular sample spaces for illustration.

2 Concepts from linear algebra

For our developments we need some standard results from linear algebra, which we state as definitions and theorems for easy of reference. Our approach is based on Queysanne, 1973, although most introductory textbooks on linear algebra should be suitable as well. The essential concept is the concept of Euclidean space, which is defined as follows.

Definition 1 *A set $\mathcal{E}$ is a q-*dimensional Euclidean space*, if, and only if, it is a q-dimensional real vector space with a positive, non-degenerate, symmetric, bilinear form.*

To refer to elements and operations in such a space we are going to use the following notation: The Abelian group operation or *sum* of two elements $\mathbf{x}, \mathbf{y} \in \mathcal{E}$ will be denoted by $\mathbf{x} \oplus \mathbf{y}$; the iterated *sum* over an index by $\bigoplus_{i=1}^{q} \mathbf{x}_i$; the neutral element with respect to $\oplus$ by $\mathbf{n}$; the inverse operation, equivalent to subtraction, by $\mathbf{x} \ominus \mathbf{y}$; and the inverse element by $\ominus\mathbf{x}$. Thus we have $\mathbf{x} \ominus \mathbf{x} = \mathbf{n}$. The external multiplication by a scalar $\alpha \in \mathbb{R}$ will be indicated by $\alpha \odot \mathbf{x}$, and we have $(-1) \odot \mathbf{x} = \ominus\mathbf{x}$, *i.e.* $\mathbf{x} \oplus ((-1) \odot \mathbf{x}) = \mathbf{x} \ominus \mathbf{x} = \mathbf{n}$. Scalar product, norm and distance will be denoted as usual by $\langle \mathbf{x}, \mathbf{y} \rangle$, $\|\mathbf{x}\|$, $\mathrm{d}(\mathbf{x}, \mathbf{y})$, using a subindex only in those cases where needed.

We know that in a real vector space $\mathcal{E}$, coefficients in any basis follow standard rules in real space, *i.e.* the Abelian group operation in $\mathcal{E}$ can be expressed as the sum of coefficients and the external multiplication defined on $\mathbb{R} \times \mathcal{E}$ as the product of a scalar with the vector of coefficients. We know also that there exists just one q-dimensional Euclidean space structure, a result derived from the following theorem.

Theorem 1 *Let be $\mathcal{E}$ a q-dimensional Euclidean space, and $\langle \cdot, \cdot \rangle$ a positive, non-degenerate, symmetric, bilinear form on it. Whatsoever this form might be, given an orthonormal basis relative to $\langle \cdot, \cdot \rangle$, for any $\mathbf{x}, \mathbf{y} \in \mathcal{E}$ whose coefficients in the given basis are $[\alpha_1, \alpha_2, \ldots, \alpha_q]$, respectively $[\beta_1, \beta_2, \ldots, \beta_q]$, it holds*

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{q} \alpha_i \beta_i, \quad \langle \mathbf{x}, \mathbf{x} \rangle = \sum_{i=1}^{q} \alpha_i^2.$$

For $q > 1$, a scalar product has infinitely many orthonormal basis associated to it (obtained *e.g.* by rotation), but we shall give a specific one in every example, which shall be denoted by $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_q\}$. Thus, whenever we consider as support space a subset of real space, any observation $\mathbf{x} \in \mathcal{E} \subseteq \mathbb{R}^p$ will be expressed either in terms of $[x_1, x_2, \ldots, x_p]$, its coefficients in the canonical basis $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ of $\mathbb{R}^p$, or as a linear combination in terms of $[\alpha_1, \alpha_2, \ldots, \alpha_q] \in \mathbb{R}^q$, its coefficients in the given basis $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_q\}$ of $\mathcal{E}$, *i.e.*

$$\begin{aligned} \mathbf{x} &= (x_1 \cdot \mathbf{u}_1) + (x_2 \cdot \mathbf{u}_2) + \cdots + (x_p \cdot \mathbf{u}_p) = \sum_{i=1}^{p} (x_i \cdot \mathbf{u}_i), \\ &= (\alpha_1 \odot \mathbf{w}_1) \oplus (\alpha_2 \odot \mathbf{w}_2) \oplus \cdots \oplus (\alpha_q \odot \mathbf{w}_q) = \bigoplus_{i=1}^{q} (\alpha_i \odot \mathbf{w}_i). \end{aligned} \tag{1}$$

Note that we distinguish the dimension p of the real space $\mathbb{R}^p$ which has the space $\mathcal{E}$ as subset, from the dimension q of the Euclidean space $\mathcal{E}$. As shall be seen, this distinction is necessary.

We can use as well matrix notation, in the understanding that rules equivalent to the usual ones apply for the sum and for the external multiplication. Thus,

$$\mathbf{x} = \bigoplus_{i=1}^{q} (\alpha_i \odot \mathbf{w}_i) = \begin{bmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_q \end{bmatrix} \odot \begin{bmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \\ \vdots \\ \mathbf{w}_q \end{bmatrix} = \vec{\alpha} \odot \mathbf{W}',$$

where $\vec{\alpha} = [\alpha_1, \alpha_2, \ldots, \alpha_q]$ and $\mathbf{W}$ stands for the row vector of vectors $[\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_q]$ which form the basis.

As mentioned, if the coefficients of two elements $\mathbf{x}, \mathbf{y} \in \mathcal{E}$ are $\vec{\alpha} = [\alpha_1, \alpha_2, \ldots, \alpha_q]$, respectively $\vec{\beta} = [\beta_1, \beta_2, \ldots, \beta_q]$, it holds

$$\mathbf{x} \oplus \mathbf{y} = \bigoplus_{i=1}^{q} ((\alpha_i + \beta_i) \odot \mathbf{w}_i) = (\vec{\alpha} + \vec{\beta}) \odot \mathbf{W}',$$

$$a \odot \mathbf{x} = \bigoplus_{i=1}^{q} ((a \cdot \alpha_i) \odot \mathbf{w}_i) = (a \cdot \vec{\alpha}) \odot \mathbf{W}',$$

where $+$ and $\cdot$ denote the usual vector sum and multiplication by a scalar in real space. In the same way we can define the inner product and the inner quotient of two elements $\mathbf{x}, \mathbf{y} \in \mathcal{E}$ as

$$\mathbf{x} \otimes \mathbf{y} = \bigoplus_{i=1}^{q} ((\alpha_i \times \beta_i) \odot \mathbf{w}_i) = (\vec{\alpha} \times \vec{\beta}) \odot \mathbf{W}',$$

$$\mathbf{x} \oslash \mathbf{y} = \bigoplus_{i=1}^{q} ((\alpha_i / \beta_i) \odot \mathbf{w}_i) = (\vec{\alpha} / \vec{\beta}) \odot \mathbf{W}', \quad \beta_i \neq 0 \quad \forall i = 1, 2, \ldots, q,$$

where $(\vec{\alpha} \times \vec{\beta})$ and $(\vec{\alpha}/\vec{\beta})$ denote, respectively, the element-by-element product and quotient of coefficients for $q \geq 1$, which reduces to the usual product and quotient of real numbers for $q = 1$. Obviously, we can define on the coefficients a vector product that leads us to a matrix of coefficients. But it would complicate both the presentation and the interpretation, and therefore we have left it out from this contribution.

The inner product and inner quotient satisfy with respect to the operations in $\mathcal{E}$ the same properties as the coefficients with respect to the operations in real space. In particular, we can consider

$$\underbrace{\mathbf{x} \otimes \mathbf{x} \otimes \cdots \otimes \mathbf{x}}_{k \quad \text{times}} = \bigoplus_{i=1}^{q} \left(\alpha_i^k \odot \mathbf{w}_i\right) = \vec{\alpha}^k \odot \mathbf{W}', \quad k \in \mathbb{N},$$

which is nothing else but a power transformation in $\mathcal{E}$ and is straightforward to generalize to any real exponent γ. This is very helpful in understanding moments, and also in defining functions of moments, like the square root. We shall write for short $\mathbf{x}^\gamma$, $\gamma \in \mathbb{R}$.

Obviously, we could define matrix operations on the coefficients, and thus on elements of $\mathcal{E}$, but this would lead us to more complex structures, which we reserve for future developments.

Finally, we will need the Lebesgue measure associated to a given basis. For its definition we need to recall two terms, namely q-volume and q-interval, which refer respectively to the generalized concept of length and interval in a q-dimensional Euclidean space $\mathcal{E}$. Thus, for $q = 1$, the 1-volume of the 1-interval is the length of an interval defined by two endpoints in a 1-dimensional Euclidean space, while for $q = 2$, the 2-volume of the 2-interval in a 2-dimensional Euclidean space is the area of a rectangle defined by the two endpoints of one of its diagonals and the parallel lines to a system of orthogonal coordinate axis, and so forth for $q > 2$.

Theorem 2 *The Lebesgue measure of a q-interval in a q-dimensional Euclidean space is the product of the coordinates of the difference vector of the two points which define the q-interval. It is precisely the q-volume of the q-interval with respect to a given orthonormal basis. Thus, for any* $\mathbf{x}, \mathbf{y} \in \mathcal{E}$ *whose coefficients in the given basis are* $[\alpha_1, \alpha_2, \ldots, \alpha_q]$, *respectively* $[\beta_1, \beta_2, \ldots, \beta_q]$, *it holds*

$$\lambda(\mathbf{x}, \mathbf{y}) = \prod_{i=1}^{q} |\beta_i - \alpha_i|.$$

Note that, for $q > 1$, the Lebesgue measure of a q-interval is the product measure obtained from the Lebesgue measure of 1-dimensional Euclidean spaces. This is consistent with the standard approach to the definition of Lebesgue measure in $\mathbb{R}^q$.

Given that subsets of an Euclidean space can be expressed either in terms of elements of the space, or in terms of the coefficients of those elements with respect to an orthonormal basis, we can state above definition in more general terms as follows.

Definition 2 *The* Lebesgue measure *of a subset* A *of a q-dimensional Euclidean space is the Lebesgue measure of the subset of* $\mathbb{R}^q$*—the space of coefficients—characterizing* A.

The meaning of the previous statements is not that for each Euclidean space $\mathcal{E}$ there exists just one vector space structure, or just one bilinear form with the above properties, or just one orthonormal basis for each bilinear form, or just one Lebesgue measure, but that, once we have chosen an Euclidean structure, the properties of the coefficients and the Lebesgue measure with respect to any orthonormal basis relative to this form will have the same expression. Thus, the easiest way, albeit not always the most intuitive one, is to apply statistics to the coefficients in such a basis and to formulate results in the representation that better reflects our understanding of the real phenomenon.

3 Introduction to statistical formulation on coefficients

This section is nothing else but a translation of elementary statistical results to random variables or vectors with a support space $\mathcal{E} \subseteq \mathbb{R}^p$ that can be structured itself as a q-dimensional real Euclidean space. For standard results in $\mathbb{R}^p$ we have used Parzen, 1960, Rohatgi, 1976, Fahrmeir and Hamerle, 1984 and Chow and Teicher, 1997, but most introductory textbooks on probability and statistics should be suitable as well. For the sake of notational simplicity, we are not going to distinguish in what follows between random variables or vectors and their realization, unless not clear from the context.

Recall that, as stated in equation (1), the elements of $\mathcal{E}$ have a dual representation: as coefficients of the canonical basis of $\mathbb{R}^p$ and as a *linear* combination of real coefficients in its own orthonormal basis. Given that the vectors of a basis are fixed, any random vector $\mathbf{x}$ defined on $\mathcal{E}$ transfers its randomness to the vector of coefficients $\vec{\alpha} = [\alpha_1, \alpha_2, \ldots, \alpha_q]$ in that basis. $\vec{\alpha}$ is thus a real random vector or, equivalently, a vector of real random coefficients. Furthermore, probability distribution and density functions are weighting functions which operate on the elements of $\mathcal{E}$. Thus, they multiply the coefficients in a given basis, and therefore, to obtain distributions on $\mathcal{E}$, we simply take distributions on those coefficients.

Definition 3 . *A* probability distribution function *of a random vector* $\mathbf{x} = \bigoplus_{i=1}^{q} (\alpha_i \odot \mathbf{w}_i)$ *on* $\mathcal{E}$ *is a joint probability distribution function* $F(\alpha_1, \alpha_2, \ldots, \alpha_q)$ *of the vector of random coefficients* $\vec{\alpha} = [\alpha_1, \alpha_2, \ldots, \alpha_q]$ *in* $\mathbb{R}^q$*. We shall write for short*

$$F_{\mathbf{x}}(\mathbf{x}) = F(\alpha_1, \alpha_2, \ldots, \alpha_q).$$

Note that, as stated in theorem 2, the Lebesgue measure in $\mathcal{E}$ is the usual Lebesgue measure in the space of coefficients, and we have

Proposition 1 *The Radon-Nykodym derivative of* $F_{\mathbf{x}}(\mathbf{x})$ *with respect to the Lebesgue measure* λ *in* $\mathcal{E}$ *is identical to the Radon-Nykodym derivative of* $F(\alpha_1, \alpha_2, \ldots, \alpha_q)$ *with respect to the Lebesgue measure* λ_q *in* $\mathbb{R}^q$*. Thus, we can write*

$$f_{\mathbf{x}}(\mathbf{x}) = \frac{\partial}{\partial \lambda} F_{\mathbf{x}}(\mathbf{x}) = \frac{\partial}{\partial \lambda_q} F(\alpha_1, \alpha_2, \ldots, \alpha_q) = f(\alpha_1, \alpha_2, \ldots, \alpha_q).$$

In most cases it is straightforward to rewrite properties of functions of random variables or vectors. For example, if $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_q\}$ is an orthonormal basis in $\mathcal{E}$, and if $\mathbf{x} = \bigoplus_{i=1}^{q} \alpha_i \odot \mathbf{w}_i$ and $\mathbf{y} = \bigoplus_{i=1}^{q} \beta_i \odot \mathbf{w}_i$ are two random vectors in $\mathcal{E}$, then

$$\mathbf{x} \oplus \mathbf{y} = \bigoplus_{i=1}^{q} \left((\alpha_i + \beta_i) \odot \mathbf{w}_i \right), \qquad \mathbf{x} \ominus \mathbf{y} = \bigoplus_{i=1}^{q} \left((\alpha_i - \beta_i) \odot \mathbf{w}_i \right)$$

and

$$\mathbf{x} \otimes \mathbf{y} = \bigoplus_{i=1}^{q} \left((\alpha_i \times \beta_i) \odot \mathbf{w}_i \right)$$

are again random vectors in $\mathcal{E}$. The same holds for

$$\mathbf{x} \oslash \mathbf{y} = \bigoplus_{i=1}^{q} \left((\alpha_i / \beta_i) \odot \mathbf{w}_i \right),$$

provided $\{\beta_i \odot \mathbf{w}_i = 0\} = \emptyset$ for all $i = 1, 2, \ldots, q$. But other cases require special attention, *e.g.* the definition of maxima and minima, or of order statistics in general, as order relationships in $\mathcal{E}$ might not transfer in a direct, intuitive way to the coefficients in a given basis. Therefore, they are not addressed in this paper.

On the coefficients we can apply standard methods to obtain the probability distribution or density of any function of random vectors as long as those functions satisfy on the coefficients the required conditions. For example, for any vector of constants $\mathbf{c} \in \mathcal{E}$, any distribution function $F_{\mathbf{x}}$ and its corresponding density function $f_{\mathbf{x}}$, it holds

$$F_{\mathbf{c} \oplus \mathbf{x}}(\mathbf{c} \oplus \mathbf{x}) = F_{\mathbf{x}}(\mathbf{x}); \qquad f_{\mathbf{c} \oplus \mathbf{x}}(\mathbf{c} \oplus \mathbf{x}) = f_{\mathbf{x}}(\mathbf{x}).$$

Now it is straightforward to define moments and centered moments whenever $\mathcal{E}$ is one dimensional ($q = 1$).

Definition 4 *Given a random variable* $\mathbf{x} = \alpha \odot \mathbf{w}$ *with support space a 1-dimensional Euclidean space* $\mathcal{E}$*, moments and centered moments in* $\mathcal{E}$ *are defined as*

$$\mathrm{E}_{\mathcal{E}}\left[\mathbf{x}^k\right] = \mathrm{E}\left[\alpha^k\right] \odot \mathbf{w}; \qquad \mathrm{M}_{\mathcal{E}}^k\left[\mathbf{x}\right] = \mathrm{E}_{\mathcal{E}}\left[\left(\mathbf{x} \ominus \mathrm{E}_{\mathcal{E}}\left[\mathbf{x}\right]\right)^k\right] = \mathrm{E}\left[\left(\alpha - \mathrm{E}\left[\alpha\right]\right)^k\right] \odot \mathbf{w},$$

whenever the corresponding moments on the coefficients exist.

Moments and centered moments in $\mathcal{E}$ are elements of the space $\mathcal{E}$, and to obtain them, just usual integration in the space of coefficients is needed. From standard properties of linear algebra we know that they depend only on the probability distribution function characterizing the random variable $\mathbf{x}$, and not on the orthonormal basis chosen. Note that

$$\mathrm{E}_{\mathcal{E}}\left[\mathbf{x}\right]=\mathrm{E}\left[\alpha\right]\odot\mathbf{w};\qquad \mathrm{M}^2_{\mathcal{E}}\left[\mathbf{x}\right]=\mathrm{E}\left[\left(\alpha-\mathrm{E}\left[\alpha\right]\right)^2\right]\odot\mathbf{w}=\mathrm{Var}\left[\alpha\right]\odot\mathbf{w}.$$

Thus, it seems reasonable to identify *characteristic elements*—like expected or central elements—of the space with moments themselves or functions thereof, and *characteristic measures* with respect to a given basis—like measures of central tendency or measures of variability—with their coefficients. Surprisingly enough, the *coefficient of variation* acquires its full meaning within this framework, as it is defined as the coefficient of the quotient of two *characteristic elements.*

In the case of $\mathcal{E}$ a q-dimensional space, the vector of expectations—which characterizes the expected element or center of a random variable—can be obtained in a similar way, as we can define

$$\mathrm{E}_{\mathcal{E}}\left[\mathbf{x}\right]=\bigoplus_{i=1}^{q}\left(\mathrm{E}\left[\alpha_i\right]\odot\mathbf{w}_i\right)=\mathrm{E}\left[\vec{\alpha}\right]\odot\mathbf{W}',$$

and analogously for centered and non-centered moments of higher order. Obviously, cross-moments, both centered and non-centered, can be defined on the coefficients easily, and they are implicitly used *e.g.* to define the bivariate normal distribution, but cross-moments in the space $\mathcal{E}$ itself are out of the scope of this paper and have been left for future extensions, as they are not always straightforward.

Note that, both for $q=1$ and for $q>1$, the usual relationship between second order centered moment and first and second non-centered moments holds, as

$$\begin{aligned}\mathrm{M}^2_{\mathcal{E}}\left[\mathbf{x}\right] &= \left(\mathrm{E}\left[\vec{\alpha}^2\right]-\left(\mathrm{E}\left[\vec{\alpha}\right]\right)^2\right)\odot\mathbf{W}'\\ &= \left(\mathrm{E}\left[\vec{\alpha}^2\right]\odot\mathbf{W}'\right)\ominus\left(\left(\mathrm{E}\left[\vec{\alpha}\right]\right)^2\odot\mathbf{W}'\right)\\ &= \mathrm{E}_{\mathcal{E}}\left[\mathbf{x}^2\right]\ominus\left(\mathrm{E}_{\mathcal{E}}\left[\mathbf{x}\right]\right)^2.\end{aligned}$$

One of the most frequently used concepts in statistics, specially in statistical inference, is the concept of independence, which is straightforward to introduce as follows.

Definition 5 *Two random variables or vectors are said to be independent if the corresponding coefficients, or vectors of coefficients, are independent.*

Note that for most purposes we will require $\mathbf{x}$ and $\mathbf{y}$ to have the same support space $\mathcal{E}$ and thus, $[\mathbf{x},\mathbf{y}]$ will have as support space the product space $\mathcal{E}\times\mathcal{E}$. Nevertheless, independence on the coefficients might refer to random variables or vectors with different support spaces, an aspect that must be taken into account when dealing with functions of independent random variables.

As can be seen, many properties transfer directly from the space of coefficients to $\mathcal{E}$ when we deal with probability distribution and density functions. But the essential thing for our purposes is, at this stage, that the study of those functions is mainly the study of some numerical characteristics or parameters associated with them, and we shall see how to do that easily in a few simple theoretical cases, which we introduce in the next section.

4 Examples

In this section, we introduce five subsets of $\mathbb{R}^q$ which can be considered as Euclidean spaces themselves, together with the basic definitions and properties that justify this assertion. Proofs are omitted, as they are straightforward, although sometimes tedious. But before we proceed, we need to point again at the duality of representation of our observations stated in equation (1). We are going to work with $\mathcal{E} \subset \mathbb{R}^q$, and we assume our observations are expressed in terms of the canonical basis $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_q\}$ of $\mathbb{R}^q$, *i.e.* we observe $\mathbf{x} \in \mathcal{E} \subset \mathbb{R}^q$ expressed as $\mathbf{x} = \sum_{i=1}^q x_i \cdot \mathbf{u}_i$. We have seen that there is an alternative representation for $\mathbf{x}$ in terms of the coefficients in an orthonormal basis of $\mathcal{E}$, *i.e.* $\mathbf{x} = \bigoplus_{i=1}^q (\alpha_i \odot \mathbf{w}_i)$ and we know we can apply standard statistical analysis to those coefficients. Our purpose is to use this fact to obtain statistical models and results and to express them in terms of $\mathbf{x} = \sum_{i=1}^q x_i \cdot \mathbf{u}_i$, *i.e.* in terms of the canonical basis of $\mathbb{R}^q$, as that is the context we are used to analyze and interpret our observations.

Example 1. Consider $\mathcal{E} = \mathbb{R}_+ \subset \mathbb{R}$. Every element $\mathbf{x} \in \mathbb{R}_+$ can be viewed as a real vector, $\mathbf{x} = x \cdot \mathbf{u}$, $x > 0$, where $\mathbf{u}$ stands for the unit vector in the real line and x for the coefficient in this basis or, as shall be seen, as an element of the Euclidean space $\mathbb{R}_+$. Although different approaches are possible, for illustration purposes we are going to use the structure given in (Pawlowsky-Glahn and Egozcue, 2001), and later used in (Mateu-Figueras, Pawlowsky-Glahn, and Martín-Fernández, 2002) and (Mateu-Figueras and Pawlowsky-Glahn, 2003), which can be summarized as follows for $\mathbf{x}, \mathbf{y} \in \mathbb{R}_+$ and $\alpha \in \mathbb{R}$.

- Abelian group operation $\oplus$, neutral element $\mathbf{n}$ and inverse element $\ominus\mathbf{x}$:

$$\mathbf{x} \oplus \mathbf{y} = xy \cdot \mathbf{u}; \quad \mathbf{n} = 1 \cdot \mathbf{u}; \quad \ominus\mathbf{x} = \frac{1}{x} \cdot \mathbf{u}.$$

- External multiplication $\odot$:

$$\alpha \odot \mathbf{x} = x^\alpha \cdot \mathbf{u}.$$

- Scalar product $\langle \cdot, \cdot \rangle$, norm $\|\cdot\|$ and distance $\mathrm{d}(\cdot, \cdot)$:

$$\langle \mathbf{x}, \mathbf{y} \rangle = \ln x \ln y; \quad \|\mathbf{x}\| = |\ln x|; \quad \mathrm{d}(\mathbf{x}, \mathbf{y}) = |\ln y - \ln x|.$$

- Unitary basis $\mathbf{w}$ and coefficient α of $\mathbf{x}$ in the given basis:

$$\mathbf{w} = e \cdot \mathbf{u}; \quad \alpha = \ln x, \ \textit{i.e.}\ \mathbf{x} = \alpha \odot \mathbf{w} = \ln x \odot (e \cdot \mathbf{u}) = e^{\ln x} \cdot \mathbf{u} = x \cdot \mathbf{u}.$$

- Internal multiplication $\otimes$ and quotient $\oslash$:

$$\mathbf{x} \otimes \mathbf{y} = (\ln x \times \ln y) \odot \mathbf{w} = (\ln x \times \ln y) \odot (e \cdot \mathbf{u}) = \exp(\ln x \times \ln y) \cdot \mathbf{u};$$

$$\mathbf{x} \oslash \mathbf{y} = \left(\frac{\ln x}{\ln y}\right) \odot \mathbf{w} = \left(\frac{\ln x}{\ln y}\right) \odot (e \cdot \mathbf{u}) = \exp\left(\frac{\ln x}{\ln y}\right) \cdot \mathbf{u}, \quad \text{for } y \neq 1.$$

- Lebesgue measure of an interval defined by the origin $\mathbf{n}$ and an arbitrary point $\mathbf{x}$:

$$\lambda = \lambda(\mathbf{n}, \mathbf{x}) = |\alpha| = |\ln x|.$$

Example 2. Consider $\mathcal{E} = \mathbb{R}_+^2 \subset \mathbb{R}^2$. Given the Euclidean space structure of $\mathbb{R}_+$, we can consider the corresponding product space structure in $\mathbb{R}_+^2$. It has been used by (Pawlowsky-Glahn, Egozcue, and Burger, 2003) to introduce an alternative model to the bivariate lognormal distribution. For $\mathbf{x} = x_1 \cdot \mathbf{u}_1 + x_2 \cdot \mathbf{u}_2$, $\mathbf{y} = y_1 \cdot \mathbf{u}_1 + y_2 \cdot \mathbf{u}_2$, $\mathbf{x}, \mathbf{y} \in \mathbb{R}_+^2$ and $\alpha \in \mathbb{R}$ it works as follows:

- Abelian group operation $\oplus$, neutral element $\mathbf{n}$ and inverse element $\ominus\mathbf{x}$:

$$\mathbf{x} \oplus \mathbf{y} = x_1 y_1 \cdot \mathbf{u}_1 + x_2 y_2 \cdot \mathbf{u}_2; \quad \mathbf{n} = 1 \cdot \mathbf{u}_1 + 1 \cdot \mathbf{u}_2; \quad \ominus\mathbf{x} = \frac{1}{x_1} \cdot \mathbf{u}_1 + \frac{1}{x_2} \cdot \mathbf{u}_2.$$

- External multiplication $\odot$:

$$\alpha \odot \mathbf{x} = x_1^\alpha \cdot \mathbf{u}_1 + x_2^\alpha \cdot \mathbf{u}_2.$$

- Scalar product $\langle \cdot, \cdot \rangle$, norm $\|\cdot\|$ and distance $\mathrm{d}(\cdot, \cdot)$:

$$\langle \mathbf{x}, \mathbf{y} \rangle = \ln x_1 \ln y_1 + \ln x_2 \ln y_2; \quad \|\mathbf{x}\| = \left[(\ln x_1)^2 + (\ln x_2)^2 \right]^{1/2};$$

$$\mathrm{d}(\mathbf{x}, \mathbf{y}) = \left[(\ln y_1 - \ln x_1)^2 + (\ln y_2 - \ln x_2)^2 \right]^{1/2}.$$

- Orthonormal basis $\{\mathbf{w}_1, \mathbf{w}_2\}$ and coefficients $[\alpha_1, \alpha_2]$ of $\mathbf{x}$ in the given basis:

$$\mathbf{w}_1 = e \cdot \mathbf{u}_1 + 1 \cdot \mathbf{u}_2, \ \mathbf{w}_2 = 1 \cdot \mathbf{u}_1 + e \cdot \mathbf{u}_2; \quad [\alpha_1, \alpha_2] = [\ln x_1, \ln x_2].$$

  Thus,

$$\begin{aligned} \mathbf{x} &= (\alpha_1 \odot \mathbf{w}_1) \oplus (\alpha_2 \odot \mathbf{w}_2) \\ &= (\ln x_1 \odot (e \cdot \mathbf{u}_1 + 1 \cdot \mathbf{u}_2)) \oplus (\ln x_2 \odot (1 \cdot \mathbf{u}_1 + e \cdot \mathbf{u}_2)) \\ &= (e^{\ln x_1} \cdot \mathbf{u}_1 + 1 \cdot \mathbf{u}_2) \oplus (1 \cdot \mathbf{u}_1 + e^{\ln x_2} \cdot \mathbf{u}_2) \\ &= (e^{\ln x_1}\, 1) \cdot \mathbf{u}_1 + (1\, e^{\ln x_2}) \cdot \mathbf{u}_2 \\ &= e^{\ln x_1} \cdot \mathbf{u}_1 + e^{\ln x_2} \cdot \mathbf{u}_2 \\ &= x_1 \cdot \mathbf{u}_1 + x_2 \cdot \mathbf{u}_2. \end{aligned}$$

- Internal multiplication $\otimes$ and quotient $\oslash$:

$$\begin{aligned} \mathbf{x} \otimes \mathbf{y} &= (\ln x_1 \times \ln y_1) \odot \mathbf{w_1} \oplus (\ln x_2 \times \ln y_2) \odot \mathbf{w_2} \\ &= \exp\{\ln x_1 \times \ln y_1\} \cdot \mathbf{u}_1 + \exp\{\ln x_2 \times \ln y_2\} \cdot \mathbf{u}_2; \end{aligned}$$

$$\begin{aligned} \mathbf{x} \oslash \mathbf{y} &= (\ln x_1 / \ln y_1) \odot \mathbf{w_1} \oplus (\ln x_2 / \ln y_2) \odot \mathbf{w_2} \\ &= \exp\{\ln x_1 / \ln y_1\} \cdot \mathbf{u}_1 + \exp\{\ln x_2 / \ln y_2\} \cdot \mathbf{u}_2; \quad \text{for } y_1 \neq 1, y_2 \neq 1 \end{aligned}$$

- Lebesgue measure of a square defined by the origin $\mathbf{n}$ and an arbitrary point $\mathbf{x}$:

$$\lambda = \lambda(\mathbf{n}, \mathbf{x}) = |\alpha_1 \alpha_2| = |\ln x_1 \ln x_2|.$$

See (Pawlowsky-Glahn, Egozcue, and Burger, 2003) for some graphical representations related to the geometry of $\mathcal{E} = \mathbb{R}_+^2$.

Example 3. Consider $\mathcal{E} = \mathcal{I} = (0, 1) \subset \mathbb{R}$. Every element $\mathbf{x} \in (0, 1)$ can be viewed as a real vector, $\mathbf{x} = x \cdot \mathbf{u}$, $0 < x < 1$, or as an element of the Euclidean space $(0, 1)$. Again, different approaches are possible, but for illustration purposes we are going to use the structure given in (Pawlowsky-Glahn and Egozcue, 2001). It can be summarized as follows for $\mathbf{x}, \mathbf{y} \in (0, 1)$ and $\alpha \in \mathbb{R}$.

- Abelian group operation $\oplus$, neutral element $\mathbf{n}$ and inverse element $\ominus \mathbf{x}$:

$$\mathbf{x} \oplus \mathbf{y} = \frac{xy}{1 - x - y + 2xy} \cdot \mathbf{u}; \quad \mathbf{n} = \frac{1}{2} \cdot \mathbf{u}; \quad \ominus \mathbf{x} = (1 - x) \cdot \mathbf{u}.$$

- External multiplication $\odot$:

$$\alpha \odot \mathbf{x} = \frac{x^\alpha}{x^\alpha + (1 - x)^\alpha} \cdot \mathbf{u}.$$

- Scalar product $\langle\cdot,\cdot\rangle$, norm $\|\cdot\|$ and distance $\mathrm{d}(\cdot,\cdot)$:

$$\langle\mathbf{x},\mathbf{y}\rangle=\ln\frac{x}{1-x}\ln\frac{y}{1-y};\quad \|\mathbf{x}\|=\left|\ln\frac{x}{1-x}\right|;\quad \mathrm{d}(\mathbf{x},\mathbf{y})=\left|\ln\frac{y}{1-y}-\ln\frac{x}{1-x}\right|.$$

- Unitary basis $\mathbf{w}$ and coefficient α of an arbitrary vector $\mathbf{x}$ in the given basis:

$$\mathbf{w}=\frac{e}{1+e}\cdot\mathbf{u};\quad \alpha=\ln\frac{x}{1-x},$$

i.e.

$$\mathbf{x}=\alpha\odot\mathbf{w}=\left(\ln\frac{x}{1-x}\right)\odot\left(\frac{e}{1+e}\cdot\mathbf{u}\right)=\frac{\exp\left(\ln\frac{x}{1-x}\right)}{1+\exp\left(\ln\frac{x}{1-x}\right)}\cdot\mathbf{u}=x\cdot\mathbf{u}.$$

- Internal multiplication $\otimes$ and quotient $\oslash$:

$$\mathbf{x}\otimes\mathbf{y}=\left(\ln\frac{x}{1-x}\,\ln\frac{y}{1-y}\right)\odot\left(\frac{e}{1+e}\cdot\mathbf{u}\right)=\frac{\exp\left(\ln\frac{x}{1-x}\ln\frac{y}{1-y}\right)}{1+\exp\left(\ln\frac{x}{1-x}\ln\frac{y}{1-y}\right)}\cdot\mathbf{u};$$

$$\mathbf{x}\oslash\mathbf{y}=\left(\frac{\ln\frac{x}{1-x}}{\ln\frac{y}{1-y}}\right)\odot\left(\frac{e}{1+e}\cdot\mathbf{u}\right)=\frac{\exp\left(\ln\frac{x}{1-x}/\ln\frac{y}{1-y}\right)}{1+\exp\left(\ln\frac{x}{1-x}/\ln\frac{y}{1-y}\right)}\cdot\mathbf{u},\qquad \text{for } y\neq 1-y.$$

- Lebesgue measure of an interval defined by the origin $\mathbf{n}$ and an arbitrary point $\mathbf{x}$:

$$\lambda=\lambda(\mathbf{n},\mathbf{x})=|\alpha|=\left|\ln\frac{x}{1-x}\right|.$$

Example 4. Consider $\mathcal{E}=(0,1)\times(0,1)=\mathcal{I}^2\subset\mathbb{R}^2$. Given the Euclidean space structure of $(0,1)$, we can consider the corresponding product space structure in $\mathcal{I}^2$. For $\mathbf{x}=x_1\cdot\mathbf{u}_1+x_2\cdot\mathbf{u}_2$, $\mathbf{y}=y_1\cdot\mathbf{u}_1+y_2\cdot\mathbf{u}_2$, $\mathbf{x},\mathbf{y}\in\mathbb{R}^2_+$ and $\alpha\in\mathbb{R}$ we obtain

- Abelian group operation $\oplus$, neutral element $\mathbf{n}$ and inverse element $\ominus\mathbf{x}$:

$$\mathbf{x}\oplus\mathbf{y}=\frac{x_1y_1}{1-x_1-y_1+2x_1y_1}\cdot\mathbf{u}_1+\frac{x_2y_2}{1-x_2-y_2+2x_2y_2}\cdot\mathbf{u}_2;$$

$$\mathbf{n}=\frac{1}{2}\cdot\mathbf{u}_1+\frac{1}{2}\cdot\mathbf{u}_2;\quad \ominus\mathbf{x}=(1-x_1)\cdot\mathbf{u}_1+(1-x_2)\cdot\mathbf{u}_2.$$

- External multiplication $\ominus$:

$$\alpha\odot\mathbf{x}=\frac{x_1^\alpha}{x_1^\alpha+(1-x_1)^\alpha}\cdot\mathbf{u}_1+\frac{x_2^\alpha}{x_2^\alpha+(1-x_2)^\alpha}\cdot\mathbf{u}_2.$$

- Scalar product $\langle\cdot,\cdot\rangle$, norm $\|\cdot\|$ and distance $\mathrm{d}(\cdot,\cdot)$:

$$\begin{aligned}
\langle\mathbf{x},\mathbf{y}\rangle &= \ln\frac{x_1}{1-x_1}\ln\frac{y_1}{1-y_1}+\ln\frac{x_2}{1-x_2}\ln\frac{y_2}{1-y_2};\\
\|\mathbf{x}\| &= \left[\left(\ln\frac{x_1}{1-x_1}\right)^2+\left(\ln\frac{x_2}{1-x_2}\right)^2\right]^{1/2};\\
\mathrm{d}(\mathbf{x},\mathbf{y}) &= \left[\left(\ln\frac{y_1}{1-y_1}-\ln\frac{x_1}{1-x_1}\right)^2+\left(\ln\frac{y_2}{1-y_2}-\ln\frac{x_2}{1-x_2}\right)^2\right]^{1/2}.
\end{aligned}$$

- Orthonormal basis $\{\mathbf{w}_1, \mathbf{w}_2\}$ and coefficients $[\alpha_1, \alpha_2]$ of an arbitrary vector $\mathbf{x}$ in the given basis:

$$\mathbf{w}_1 = \left(\frac{e}{1+e}\right) \cdot \mathbf{u}_1 + \frac{1}{2} \cdot \mathbf{u}_2,\ \mathbf{w}_2 = \frac{1}{2} \cdot \mathbf{u}_1 + \left(\frac{e}{1+e}\right) \cdot \mathbf{u}_2; \quad [\alpha_1, \alpha_2] = \left[\ln \frac{x_1}{1-x_1}, \ln \frac{x_2}{1-x_2}\right].$$

Thus,

$$\begin{aligned}
\mathbf{x} &= (\alpha_1 \odot \mathbf{w}_1) \oplus (\alpha_2 \odot \mathbf{w}_2) \\
&= \left(\ln \frac{x_1}{1-x_1} \odot \left(\frac{e}{1+e} \cdot \mathbf{u}_1 + \frac{1}{2} \cdot \mathbf{u}_2\right)\right) \oplus \left(\ln \frac{x_2}{1-x_2} \odot \left(\frac{1}{2} \cdot \mathbf{u}_1 + \frac{e}{1+e} \cdot \mathbf{u}_2\right)\right) \\
&= \frac{\exp\left(\ln \frac{x_1}{1-x_1}\right)}{1+\exp\left(\ln \frac{x_1}{1-x_1}\right)} \cdot \mathbf{u}_1 + \frac{\exp\left(\ln \frac{x_2}{1-x_2}\right)}{1+\exp\left(\ln \frac{x_2}{1-x_2}\right)} \cdot \mathbf{u}_2 \\
&= x_1 \cdot \mathbf{u}_1 + x_2 \cdot \mathbf{u}_2
\end{aligned}$$

- Internal multiplication $\otimes$ and quotient $\oslash$:

$$\begin{aligned}
\mathbf{x} \otimes \mathbf{y} &= \left(\ln \frac{x_1}{1-x_1} \times \ln \frac{y_1}{1-y_1}\right) \odot \mathbf{w_1} \oplus \left(\ln \frac{x_2}{1-x_2} \times \ln \frac{y_2}{1-y_2}\right) \odot \mathbf{w_2} \\
&= \frac{\exp\left(\ln \frac{x_1}{1-x_1} \times \ln \frac{y_1}{1-y_1}\right)}{1+\exp\left(\ln \frac{x_1}{1-x_1} \times \ln \frac{y_1}{1-y_1}\right)} \cdot \mathbf{u}_1 + \frac{\exp\left(\ln \frac{x_2}{1-x_2} \times \ln \frac{y_2}{1-y_2}\right)}{1+\exp\left(\ln \frac{x_2}{1-x_2} \times \ln \frac{y_2}{1-y_2}\right)} \cdot \mathbf{u}_2;
\end{aligned}$$

$$\begin{aligned}
\mathbf{x} \oslash \mathbf{y} &= \left(\ln \frac{x_1}{1-x_1} / \ln \frac{y_1}{1-y_1}\right) \odot \mathbf{w_1} \oplus \left(\ln \frac{x_2}{1-x_2} / \ln \frac{y_2}{1-y_2}\right) \odot \mathbf{w_2} \\
&= \frac{\exp\left(\ln \frac{x_1}{1-x_1} / \ln \frac{y_1}{1-y_1}\right)}{1+\exp\left(\ln \frac{x_1}{1-x_1} / \ln \frac{y_1}{1-y_1}\right)} \cdot \mathbf{u}_1 + \frac{\exp\left(\ln \frac{x_2}{1-x_2} / \ln \frac{y_2}{1-y_2}\right)}{1+\exp\left(\ln \frac{x_2}{1-x_2} / \ln \frac{y_2}{1-y_2}\right)} \cdot \mathbf{u}_2,
\end{aligned}$$

provided $y_1 \neq 1 - y_1$ and $y_2 \neq 1 - y_2$.

- Lebesgue measure of a rectangle defined by the origin $\mathbf{n}$ and an arbitrary point $\mathbf{x}$:

$$\lambda = \lambda(\mathbf{n}, \mathbf{x}) = |\alpha_1 \alpha_2| = \left|\ln \frac{x_1}{1-x_1} \ln \frac{x_2}{1-x_2}\right|.$$

Example 5. Consider $\mathcal{E}$ to be the 3-part unit simplex $\mathcal{S}^3$, *i.e.*

$$\mathcal{E} = \mathcal{S}^3 = \{\mathbf{x} = x_1 \cdot \mathbf{u}_1 + x_2 \cdot \mathbf{u}_2 + x_3 \cdot \mathbf{u}_3 \mid x_i > 0, i = 1,2,3; x_1 + x_2 + x_3 = 1\} \subset \mathbb{R}^3,$$

and let $\mathcal{C}[\cdot]$ denote the closure operator defined for any $\mathbf{z} = z_1 \cdot \mathbf{u}_1 + z_2 \cdot \mathbf{u}_2 + z_3 \cdot \mathbf{u}_3 \in \mathbb{R}^3_+$ as

$$\mathcal{C}[\mathbf{z}] = \mathcal{C}[z_1 \cdot \mathbf{u}_1 + z_2 \cdot \mathbf{u}_2 + z_3 \cdot \mathbf{u}_3] = \frac{z_1}{z_1+z_2+z_3} \cdot \mathbf{u}_1 + \frac{z_3}{z_1+z_2+z_3} \cdot \mathbf{u}_2 + \frac{z_3}{z_1+z_2+z_3} \cdot \mathbf{u}_3.$$

It is well known that $\mathcal{S}^D$, $D \geq 2$ is an Euclidean space, and its properties have been extensively used (Aitchison, 2002; Aitchison, Barceló-Vidal, Egozcue, and Pawlowsky-Glahn, 2002; Billheimer, Guttorp, and Fagan, 2001; Pawlowsky-Glahn and Egozcue, 2001, 2002; Egozcue, Pawlowsky-Glahn, Mateu-Figueras, and Barceló-Vidal, 2003). Therefore, here we only synthesize the main characteristics using the same scheme as in the previous examples. To simplify the presentation we shall use vector and matrix notation, whenever needed; *i.e.*

$$\mathbf{x} = x_1 \cdot \mathbf{u}_1 + x_2 \cdot \mathbf{u}_2 + x_3 \cdot \mathbf{u}_3 = [x_1, x_2, x_3] \cdot \begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \mathbf{u}_3 \end{pmatrix} = \vec{x} \cdot \mathbf{U}',$$

and, given that the closure operation affects only the coefficients, we shall write, indistinctly,

$$\mathcal{C}\left[\mathbf{z}\right]=\mathcal{C}\left[z_1\cdot\mathbf{u}_1+z_2\cdot\mathbf{u}_2+z_3\cdot\mathbf{u}_3\right]=\mathcal{C}\left[z_1,z_2,z_3\right]\cdot\mathbf{U}'=\mathcal{C}\left[\vec{z}\right]\cdot\mathbf{U}'.$$

- Abelian group operation $\oplus$, neutral element $\mathbf{n}$ and inverse element $\ominus\mathbf{x}$:

$$\mathbf{x}\oplus\mathbf{y}=\mathcal{C}\left[x_1y_1\cdot\mathbf{u}_1+x_2y_2\cdot\mathbf{u}_2+x_3y_3\cdot\mathbf{u}_3\right]=\mathcal{C}\left[x_1y_1,x_2y_2,x_3y_3\right]\cdot\mathbf{U}';$$

$$\mathbf{n}=\mathcal{C}\left[1,1,1\right]\cdot\mathbf{U}'=\left[\frac{1}{3},\frac{1}{3},\frac{1}{3}\right]\cdot\mathbf{U}';\quad \ominus\mathbf{x}=\mathcal{C}\left[\frac{1}{x_1},\frac{1}{x_2},\frac{1}{x_3}\right]\cdot\mathbf{U}'.$$

- External multiplication $\odot$:

$$\alpha\odot x=\mathcal{C}\left[x_1^{\alpha},x_2^{\alpha},x_3^{\alpha}\right]\cdot\mathbf{U}'.$$

- Scalar product $\langle\cdot,\cdot\rangle$, norm $\|\cdot\|$ and distance $\mathrm{d}\left(\cdot,\cdot\right)$:

$$\langle x,y\rangle=\frac{1}{3}\left(\ln\frac{x_1}{x_2}\ln\frac{y_1}{y_2}+\ln\frac{x_1}{x_3}\ln\frac{y_1}{y_3}+\ln\frac{x_2}{x_3}\ln\frac{y_2}{y_3}\right);$$

$$\|x\|=\frac{1}{\sqrt{3}}\left[\left(\ln\frac{x_1}{x_2}\right)^2+\left(\ln\frac{x_1}{x_3}\right)^2+\left(\ln\frac{x_2}{x_3}\right)^2\right]^{1/2};$$

$$\mathrm{d}\left(x,y\right)=\frac{1}{\sqrt{3}}\left[\left(\ln\frac{x_1}{x_2}-\ln\frac{y_1}{y_2}\right)^2+\left(\ln\frac{x_1}{x_3}-\ln\frac{y_1}{y_3}\right)^2+\left(\ln\frac{x_2}{x_3}-\ln\frac{y_2}{y_3}\right)^2\right]^{1/2}.$$

- Orthonormal basis $\{\mathbf{w}_1,\mathbf{w}_2\}$ and coefficients $[\alpha_1,\alpha_2]$ of an arbitrary vector $\mathbf{x}$ in the given basis:

$$\mathbf{w}_1=\mathcal{C}\left[\exp\frac{1}{\sqrt{2}},\exp\frac{-1}{\sqrt{2}},1\right]\cdot\mathbf{U}',\quad \mathbf{w}_2=\mathcal{C}\left[\exp\sqrt{\frac{3}{2}},\exp\sqrt{\frac{3}{2}},1\right]\cdot\mathbf{U}';$$

$$[\alpha_1,\alpha_2]=\left[\frac{1}{\sqrt{2}}\ln\frac{x_1}{x_2},\frac{1}{\sqrt{6}}\ln\frac{x_1x_2}{x_3x_3}\right],$$

i.e.

$$\begin{aligned}\mathbf{x}&=(\alpha_1\odot\mathbf{w}_1)\oplus(\alpha_2\odot\mathbf{w}_2)\\&=\mathcal{C}\left[\exp\frac{\alpha_1}{\sqrt{2}}\cdot\mathbf{u}_1+\exp\frac{-\alpha_1}{\sqrt{2}}\cdot\mathbf{u}_2+\mathbf{u}_3\right]\oplus\mathcal{C}\left[\exp\frac{\sqrt{3}\alpha_2}{\sqrt{2}}\cdot\mathbf{u}_1+\exp\frac{\sqrt{3}\alpha_2}{\sqrt{2}}\cdot\mathbf{u}_2+\mathbf{u}_3\right]\\&=\mathcal{C}\left[\exp\frac{\alpha_1}{\sqrt{2}}\exp\frac{\sqrt{3}\alpha_2}{\sqrt{2}}\cdot\mathbf{u}_1+\exp\frac{-\alpha_1}{\sqrt{2}}\exp\frac{\sqrt{3}\alpha_2}{\sqrt{2}}\cdot\mathbf{u}_2+\mathbf{u}_3\right].\end{aligned}$$

But

$$\exp\frac{\alpha_1}{\sqrt{2}}\exp\frac{\sqrt{3}\alpha_2}{\sqrt{2}}=\exp\frac{1}{2}\ln\frac{x_1}{x_2}\exp\frac{3}{6}\ln\frac{x_1x_2}{x_3x_3}=\sqrt{\frac{x_1}{x_2}}\sqrt{\frac{x_1x_2}{x_3x_3}}=\frac{x_1}{x_3},$$

and analogously

$$\exp\frac{-\alpha_1}{\sqrt{2}}\exp\frac{\sqrt{3}\alpha_2}{\sqrt{2}}=\frac{x_2}{x_3},$$

leading to

$$\mathbf{x}=\mathcal{C}\left[\frac{x_1}{x_3}\cdot\mathbf{u}_1+\frac{x_2}{x_3}\cdot\mathbf{u}_2+\mathbf{u}_3\right]=x_1\cdot\mathbf{u}_1+x_2\cdot\mathbf{u}_2+x_3\cdot\mathbf{u}_3.$$

- Internal multiplication $\otimes$ and quotient $\oslash$:

$$\begin{aligned}
\mathbf{x}\otimes\mathbf{y} &= \left(\frac{1}{\sqrt{2}}\ln\frac{x_1}{x_2}\times\frac{1}{\sqrt{2}}\ln\frac{y_1}{y_2}\right)\odot\mathbf{w_1}\oplus\left(\frac{1}{\sqrt{6}}\ln\frac{x_1x_2}{x_3x_3}\times\frac{1}{\sqrt{6}}\ln\frac{y_1y_2}{y_3y_3}\right)\odot\mathbf{w_2}\\
&= \mathcal{C}\left[\beta_1\cdot\mathbf{u}_1+\beta_2\cdot\mathbf{u}_2+\beta_3\cdot\mathbf{u}_3\right],
\end{aligned}$$

where

$$\begin{aligned}
\beta_1 &= \exp\left(\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\ln\frac{x_1}{x_2}\times\frac{1}{\sqrt{2}}\ln\frac{y_1}{y_2}\right)\right)\exp\left(\sqrt{\frac{3}{2}}\left(\frac{1}{\sqrt{6}}\ln\frac{x_1x_2}{x_3x_3}\times\frac{1}{\sqrt{6}}\ln\frac{y_1y_2}{y_3y_3}\right)\right)\\
&= \exp\left(\frac{1}{2\sqrt{2}}\left(\ln\frac{x_1}{x_2}\times\ln\frac{y_1}{y_2}\right)\right)\exp\left(\frac{1}{2\sqrt{6}}\left(\ln\frac{x_1x_2}{x_3x_3}\times\ln\frac{y_1y_2}{y_3y_3}\right)\right)\\
\beta_2 &= \exp\left(\frac{-1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\ln\frac{x_1}{x_2}\times\frac{1}{\sqrt{2}}\ln\frac{y_1}{y_2}\right)\right)\exp\left(\sqrt{\frac{3}{2}}\left(\frac{1}{\sqrt{6}}\ln\frac{x_1x_2}{x_3x_3}\times\frac{1}{\sqrt{6}}\ln\frac{y_1y_2}{y_3y_3}\right)\right)\\
&= \exp\left(\frac{-1}{2\sqrt{2}}\left(\ln\frac{x_1}{x_2}\times\ln\frac{y_1}{y_2}\right)\right)\exp\left(\frac{1}{2\sqrt{6}}\left(\ln\frac{x_1x_2}{x_3x_3}\times\ln\frac{y_1y_2}{y_3y_3}\right)\right)\\
\beta_3 &= 1. \qquad (2)
\end{aligned}$$

$$\begin{aligned}
\mathbf{x}\oslash\mathbf{y} &= \left(\frac{1}{\sqrt{2}}\ln\frac{x_1}{x_2}/\frac{1}{\sqrt{2}}\ln\frac{y_1}{y_2}\right)\odot\mathbf{w_1}\oplus\left(\frac{1}{\sqrt{6}}\ln\frac{x_1x_2}{x_3x_3}/\frac{1}{\sqrt{6}}\ln\frac{y_1y_2}{y_3y_3}\right)\odot\mathbf{w_2}\\
&= \mathcal{C}\left[\gamma_1\cdot\mathbf{u}_1+\gamma_2\cdot\mathbf{u}_2+\gamma_3\cdot\mathbf{u}_3\right],
\end{aligned}$$

where

$$\begin{aligned}
\gamma_1 &= \exp\left(\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\ln\frac{x_1}{x_2}/\frac{1}{\sqrt{2}}\ln\frac{y_1}{y_2}\right)\right)\exp\left(\sqrt{\frac{3}{2}}\left(\frac{1}{\sqrt{6}}\ln\frac{x_1x_2}{x_3x_3}/\frac{1}{\sqrt{6}}\ln\frac{y_1y_2}{y_3y_3}\right)\right)\\
&= \exp\left(\frac{1}{2\sqrt{2}}\left(\ln\frac{x_1}{x_2}/\ln\frac{y_1}{y_2}\right)\right)\exp\left(\frac{1}{2\sqrt{6}}\left(\ln\frac{x_1x_2}{x_3x_3}/\ln\frac{y_1y_2}{y_3y_3}\right)\right)\\
\gamma_2 &= \exp\left(\frac{-1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\ln\frac{x_1}{x_2}/\frac{1}{\sqrt{2}}\ln\frac{y_1}{y_2}\right)\right)\exp\left(\sqrt{\frac{3}{2}}\left(\frac{1}{\sqrt{6}}\ln\frac{x_1x_2}{x_3x_3}/\frac{1}{\sqrt{6}}\ln\frac{y_1y_2}{y_3y_3}\right)\right)\\
&= \exp\left(\frac{-1}{2\sqrt{2}}\left(\ln\frac{x_1}{x_2}/\ln\frac{y_1}{y_2}\right)\right)\exp\left(\frac{1}{2\sqrt{6}}\left(\ln\frac{x_1x_2}{x_3x_3}/\ln\frac{y_1y_2}{y_3y_3}\right)\right)\\
\gamma_3 &= 1, \qquad (3)
\end{aligned}$$

provided $y_1\neq y_2$ and $y_1y_2\neq y_3^2$.

- Lebesgue measure of a square defined by the origin $\mathbf{n}$ and an arbitrary point $\mathbf{x}$:

$$\lambda=\lambda(\mathbf{n},\mathbf{x})=|\alpha_1\alpha_2|=\left|\frac{1}{\sqrt{2}}\ln\frac{x_1}{x_2}\frac{1}{\sqrt{6}}\ln\frac{x_1x_2}{x_3x_3}\right|.$$

For a graphical representation of multiple geometric elements, see the references cited at the beginning of this example.

It is important to note that, like in every Euclidean space, the orthonormal basis given is not unique. In fact, there are infinitely many, like in $\mathbb{R}_+^2$ or in $\mathcal{I}^2$. With the simplex we have an additional difficulty: it is not straightforward to determine which one is the most appropriate orthonormal basis we can choose to solve a specific problem. Some aspects related to it have been already addresses in (Pawlowsky-Glahn and Egozcue, 2001). The important point is that, once we have chosen an appropriate orthonormal basis, all the results developed here are valid taking the appropriate coefficients with respect to it.

5 The normal distribution and its properties

In the above examples we have either a one or a two dimensional sample space $\mathcal{E}$. Therefore, in what follows, we are going to use either the uni- or the bivariate normal distribution on the coefficients. From standard theory, we know that for this distribution we can obtain *e.g.* predictive intervals or regions, isoprobability intervals or curves for known parameters. If we do not know the parameters, we can estimate them from a sample, and we can look for whatever properties we might be interested in. Using above theory, we can transfer all those items to $\mathcal{E}$. The question is how to interpret these results. To have a glimpse at the way to proceed, let us consider first some general properties and then each example separately.

5.1 The univariate normal on a one dimensional Euclidean space

If $\mathcal{E}$ is a one dimensional Euclidean space, any random variable $\mathbf{x}$ with support space $\mathcal{E}$ can be expressed as $\mathbf{x} = \alpha \odot \mathbf{w}$, where α is a real random variable and $\mathbf{w}$ is a basis of $\mathcal{E}$. Thus, we can define a normal distribution on $\mathcal{E}$ and determine its properties. Proofs are not required, as they follow from standard statistical theory in real space and basic linear algebra.

Definition 6 *A random variable* $\mathbf{x}$ *is said to follow a standard normal distribution on* $\mathcal{E}$ *if its density function is*

$$f_{\mathbf{x}}(\mathbf{x}) = f(\alpha) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\alpha^2}{2}\right), \quad \alpha \in \mathbb{R}.$$

$\mathbf{x}$ *is said to follow a normal distribution on* $\mathcal{E}$ *with parameters* μ *and* σ*, if its density function is*

$$f_{\mathbf{x}}(\mathbf{x}) = f(\alpha) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(\alpha-\mu)^2}{2\sigma^2}\right), \quad \alpha \in \mathbb{R}.$$

We shall write for short $\mathbf{x} \sim \mathcal{N}_{\mathcal{E}}(0,1)$*, respectively* $\mathbf{x} \sim \mathcal{N}_{\mathcal{E}}(\mu, \sigma^2)$*.*

Proposition 2 *If* $\mathbf{x} \sim \mathcal{N}_{\mathcal{E}}(\mu, \sigma^2)$*, then* $\frac{1}{\sigma} \odot (\mathbf{x} \ominus (\mu \odot \mathbf{w})) \sim \mathcal{N}_{\mathcal{E}}(0,1)$*.*

This property is straightforward if we take into account that

$$\frac{1}{\sigma} \odot (\mathbf{x} \ominus (\mu \odot \mathbf{w})) = \frac{1}{\sigma} \odot ((\alpha - \mu) \odot \mathbf{w}) = \left(\frac{\alpha-\mu}{\sigma}\right) \odot \mathbf{w}.$$

Proposition 3 *The normal distribution on* $\mathcal{E}$ *is stable,* e.g.*, given two independent identically distributed random variables* $\mathbf{x}_1 \sim \mathcal{N}_{\mathcal{E}}(\mu, \sigma^2)$ *and* $\mathbf{x}_2 \sim \mathcal{N}_{\mathcal{E}}(\mu, \sigma^2)$*, and given two real constants* $a_1, a_2 > 0$*, a real number* $b > 0$ *and an element* $\mathbf{c} \in \mathcal{E}$ *can be found such that*

$$\mathbf{x}_3 = \frac{1}{b} \odot (a_1 \odot \mathbf{x}_1 \oplus a_2 \odot \mathbf{x}_2 \ominus \mathbf{c}) \sim \mathcal{N}_{\mathcal{E}}(\mu, \sigma^2).$$

Proposition 4 *The moments of a random variable* $\mathbf{x} \sim \mathcal{N}_{\mathcal{E}}(\mu, \sigma^2)$ *satisfy the following properties:*

(a) $\mathrm{E}_{\mathcal{E}}[\mathbf{x}] = \mu \odot \mathbf{w}$*;*

(b) $\mathrm{E}_{\mathcal{E}}\left[\mathbf{x}^2\right] = (\sigma^2 + \mu^2) \odot \mathbf{w} = \sigma^2 \odot \mathbf{w} \oplus \mu^2 \odot \mathbf{w}$*;*

(c) $\mathrm{M}^2_{\mathcal{E}}[\mathbf{x}] = \sigma^2 \odot \mathbf{w}$*;*

(d) $\mathrm{M}^k_{\mathcal{E}}[\mathbf{x}] = 0 \odot \mathbf{w}, \quad \forall k = 2n+1, n \in \mathbb{N}$*;*

(e) $\mathrm{M}_{\mathcal{E}}^{k}\left[\mathbf{x}\right] = \left((k-1)(k-3)\cdots 3\cdot 1\cdot \sigma^{k}\right)\odot \mathbf{w}, \quad \forall k = 2n, n \in \mathbb{N}.$

Proposition 5 *For $\mathbf{x}_1$, $\mathbf{x}_2$, ..., $\mathbf{x}_n$ independent random variables on $\mathcal{E}$, $\mathbf{x}_k \sim \mathcal{N}_{\mathcal{E}}(\mu_k, \sigma_k^2)$, $k = 1, 2, \ldots, n$,*

$$\mathbf{s}_n = \bigoplus_{k=1}^{n} \mathbf{x}_k \sim \mathcal{N}_{\mathcal{E}}\left(\sum_{k=1}^{n}\mu_k, \sum_{k=1}^{n}\sigma_k^2\right).$$

Furthermore, for $\mathbf{x}_k$ independent identically distributed, $\mathbf{x}_k \sim \mathcal{N}_{\mathcal{E}}(\mu, \sigma^2)$, $k = 1, 2, \ldots, n$,

$$\mathbf{s}_n \sim \mathcal{N}_{\mathcal{E}}\left(n\mu, n\sigma^2\right) \qquad \textit{and} \qquad \frac{1}{n}\odot \mathbf{s}_n \sim \mathcal{N}_{\mathcal{E}}\left(\mu, \frac{\sigma^2}{n}\right).$$

Finally, for $\mathbf{x}_k \sim \mathcal{N}_{\mathcal{E}}(0,1)$, $k = 1, 2, \ldots, n$,

$$\frac{1}{\sqrt{n}}\odot \mathbf{s}_n \sim \mathcal{N}_{\mathcal{E}}\left(0,1\right).$$

Proposition 6 *If $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$, with $\mathbf{x}_k = \alpha_k \odot \mathbf{w}$, $k = 1, 2, \ldots, n$, is a sample from a $\mathcal{N}_{\mathcal{E}}(\mu, \sigma^2)$, where both μ and σ^2 are unknown, the maximum likelihood estimates of μ and σ^2 are, respectively,*

$$\hat{\mu} = \frac{1}{n}\sum_{k=1}^{n}\alpha_k = \bar{\alpha} \qquad \textit{and} \qquad \hat{\sigma}^2 = \frac{1}{n}\sum_{k=1}^{n}\left(\alpha_k - \bar{\alpha}\right)^2.$$

Confidence intervals for the estimated parameters will be consequently the usual intervals, and to obtain $(1-\gamma)$ confidence intervals around the expected value $\mathrm{E}_{\mathcal{E}}\left[\mathbf{x}\right]$ we just need to apply the extremes to the basis, resulting for σ known in

$$\left[\left(\hat{\mu} - z_{\gamma/2}\frac{\sigma}{\sqrt{n}}\right)\odot \mathbf{w}, \left(\hat{\mu} + z_{\gamma/2}\frac{\sigma}{\sqrt{n}}\right)\odot \mathbf{w}\right]$$
$$= \left[\hat{\mu}\odot\mathbf{w} \ominus \left(z_{\gamma/2}\frac{\sigma}{\sqrt{n}}\right)\odot \mathbf{w}, \hat{\mu}\odot\mathbf{w} \oplus \left(z_{\gamma/2}\frac{\sigma}{\sqrt{n}}\right)\odot \mathbf{w}\right],$$

where $z_{\gamma/2}$ stands for the $1-\gamma/2$ quantile of the standard normal distribution in $\mathbb{R}$. This interval has shortest length in $\mathcal{E}$.

Analogously, for σ unknown we obtain

$$\left[\left(\hat{\mu} - \mathtt{t}_{n,\gamma/2}\frac{\sigma}{\sqrt{n}}\right)\odot \mathbf{w}, \left(\hat{\mu} + \mathtt{t}_{n,\gamma/2}\frac{\sigma}{\sqrt{n}}\right)\odot \mathbf{w}\right]$$
$$= \left[\hat{\mu}\odot\mathbf{w} \ominus \left(\mathtt{t}_{n,\gamma/2}\frac{\sigma}{\sqrt{n}}\right)\odot \mathbf{w}, \hat{\mu}\odot\mathbf{w} \oplus \left(\mathtt{t}_{n,\gamma/2}\frac{\sigma}{\sqrt{n}}\right)\odot \mathbf{w}\right],$$

where $\mathtt{t}_{n,\gamma/2}$ stands for the $1-\gamma/2$ quantile of Students $\mathtt{t}$ distribution in $\mathbb{R}$. This interval has shortest expected length in $\mathcal{E}$.

Obviously, following the same scheme, we could proceed now to define confidence intervals around the second order central moment using usual confidence intervals around the variance, but their usefulness is an open question.

5.2 The bivariate normal on a two dimensional Euclidean space

Assume now $\mathcal{E}$ is a two dimensional Euclidean space. Thus, any random vector $\mathbf{x}$ with support space $\mathcal{E}$ can be expressed as $\mathbf{x} = \alpha_1 \odot \mathbf{w}_1 \oplus \alpha_2 \odot \mathbf{w}_2$, and $[\alpha_1, \alpha_2]$ will be a real random vector. Thus, we can define a bivariate normal distribution on $\mathcal{E}$.

Definition 7 *A random vector* $\mathbf{x}$ *is said to follow a bivariate normal distribution on* $\mathcal{E}$ *with parameters* μ_1*,* μ_2*,* σ_1*,* σ_2 *and* ρ *if its density function is*

$$f_{\mathbf{x}}(\mathbf{x}) = f(\alpha_1, \alpha_2) = \frac{1}{2\pi\sigma_1\sigma_2(1-\rho^2)^{1/2}} \cdot \exp\left(-\frac{A}{2(1-\rho)^2}\right)$$

where

$$A = \left[\frac{(\alpha_1-\mu_1)^2}{\sigma_1^2} - \frac{2\rho(\alpha_1-\mu_1)(\alpha_2-\mu_2)}{\sigma_1\sigma_2} + \frac{(\alpha_2-\mu_2)^2}{\sigma_2^2}\right],$$

$\alpha_1, \alpha_2, \mu_1, \mu_2 \in \mathbb{R}; \sigma_1, \sigma_2 > 0; |\rho| < 1$. *We shall write for short* $\mathbf{x} \sim \mathcal{N}_{\mathcal{E}}(\mu_1, \mu_2; \sigma_1^2, \sigma_2^2, \rho)$.

To illustrate how working on coefficients can be straightforward from a mathematical point of view, but difficult to interpret in some cases, we are going to use a property of independence of marginals and the moments as defined above.

Proposition 7 *For* $\mu_1 = \mu_2 = 0$*,* $\rho = 0$ *and* $\sigma_1 = \sigma_2 = \sigma$*, the real random variables* α_1 *and* α_2 *are independent. If each component of* $\mathbf{x}$ *is uniquely associated to a coefficient, then the components of* $\mathbf{x}$ *themselves are independent.*

Note the restriction on the definition of independence of the components of $\mathbf{x}$, which is due to the fact that this result cannot be transferred directly to the components in general, as shall be seen in example 5.

Now, using the fact that in the space of coefficients, marginals of a real random vector following a multivariate normal distribution follow also normal distributions, we can find the vectors of moments as defined above.

Proposition 8 *The moments of a random vector* $\mathbf{x} \sim \mathcal{N}_{\mathcal{E}}(\mu_1, \mu_2; \sigma_1^2, \sigma_2^2, \rho)$ *satisfy the following properties:*

(a) $\mathrm{E}_{\mathcal{E}}[\mathbf{x}] = [\mu_1, \mu_2] \odot \mathbf{W}'$;

(b) $\mathrm{E}_{\mathcal{E}}[\mathbf{x}^2] = [\sigma_1^2 + \mu_1^2, \sigma_2^2 + \mu_2^2] \odot \mathbf{W}' = [\sigma_1^2, \sigma_2^2] \odot \mathbf{W}' \oplus [\mu_1^2, \mu_2^2] \odot \mathbf{W}'$;

(c) $\mathrm{M}^2_{\mathcal{E}}[\mathbf{x}] = [\sigma_1^2, \sigma_2^2] \odot \mathbf{W}'$;

(d) $\mathrm{M}^k_{\mathcal{E}}[\mathbf{x}] = [0, 0] \odot \mathbf{W}', \quad \forall k = 2n+1, n \in \mathbb{N}$;

(e) $\mathrm{M}^k_{\mathcal{E}}[\mathbf{x}] = \left[\left((k-1)(k-3)\cdots 3 \cdot 1 \cdot \sigma_1^k\right), \left((k-1)(k-3)\cdots 3 \cdot 1 \cdot \sigma_2^k\right)\right] \odot \mathbf{W}', \quad \forall k = 2n, n \in \mathbb{N}$,

where $\mathbf{W} = [\mathbf{w}_1, \mathbf{w}_2]$ *is the orthonormal basis of* $\mathcal{E}$ *used for reference.*

The parameter ρ of the bivariate normal distribution can be certainly interpreted as a measure of linear dependence between the real random coefficients α_1 and α_2. Again, only in the case of a one-to-one relationship between one components and one coefficient we shall be able to use it also as a measure of linear dependence between the components, but this is not a general property.

Concerning estimation of parameters, the usual rules apply, as they are defined in terms of the coefficients. It is only in the case of seeking estimates of moments in $\mathcal{E}$ that we will apply them to the basis of the space.

5.3 Examples

In this section we are going to show on the above introduced sample spaces how working on coefficients is done in practice. Some of the properties sound strange and are probably of little interest in practice, but they have been included for illustration purposes.

Example 1. Let us consider $\mathcal{E} = \mathbb{R}_+$, which is a one dimensional Euclidean space. We start from a given unitary basis $\mathbf{w} = e \cdot \mathbf{u}$. The expression of a random variable in terms of this basis is

$$\mathbf{x} = \alpha \odot \mathbf{w} = \ln x \odot \mathbf{w} = e^{\ln x} \cdot \mathbf{u} = x \cdot \mathbf{u}.$$

The principal aspects of the normal distribution on $\mathbb{R}_+$, together with some graphical illustrations, can be found in (Mateu-Figueras, Pawlowsky-Glahn, and Martín-Fernández, 2002; Mateu-Figueras and Pawlowsky-Glahn, 2003).

Definition 8 *A random variable* $\mathbf{x}$ *is said to follow a standard normal distribution on* $\mathbb{R}_+$ *if its density function is*

$$f_{\mathbf{x}}(\mathbf{x}) = f(\ln x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(\ln x)^2}{2}\right), \quad 0 < x.$$

$\mathbf{x}$ *is said to follow a normal distribution on* $\mathbb{R}_+$ *with parameters* μ *and* σ*, if its density function is*

$$f_{\mathbf{x}}(\mathbf{x}) = f(\ln x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right), \quad 0 < x.$$

We shall write for short $\mathbf{x} \sim \mathcal{N}_+(0,1)$*, respectively* $\mathbf{x} \sim \mathcal{N}_+(\mu, \sigma^2)$.

Proposition 9 *If* $\mathbf{x} \sim \mathcal{N}_+(\mu, \sigma^2)$*, then* $\left(x^{1/\sigma} \cdot e^{-\mu/\sigma}\right) \cdot \mathbf{u} \sim \mathcal{N}_+(0,1)$.

Proposition 10 *The normal distribution on* $\mathbb{R}_+$ *is stable,* e.g.*, given two independent identically distributed random variables* $\mathbf{x}_1 \sim \mathcal{N}_+(\mu, \sigma^2)$ *and* $\mathbf{x}_2 \sim \mathcal{N}_+(\mu, \sigma^2)$*, and given two real constants* $a_1, a_2 > 0$*, a real number* $b > 0$ *and an element* $\mathbf{c} = c \cdot \mathbf{u} \in \mathbb{R}_+$ *can be found such that*

$$\mathbf{x}_3 = \left(\frac{x_1^{a_1} \cdot x_2^{a_2}}{c}\right)^{1/b} \cdot \mathbf{u} \sim \mathcal{N}_+(\mu, \sigma^2).$$

Proposition 11 *The moments of a random variable* $\mathbf{x} \sim \mathcal{N}_+(\mu, \sigma^2)$ *satisfy the following properties:*

(a) $\mathrm{E}_+[\mathbf{x}] = e^{\mu} \cdot \mathbf{u}$*;*

(b) $\mathrm{E}_+\left[\mathbf{x}^2\right] = e^{\sigma^2+\mu^2} \cdot \mathbf{u}$*;*

(c) $\mathrm{M}_+^2[\mathbf{x}] = e^{\sigma^2} \cdot \mathbf{u}$*;*

(d) $\mathrm{M}_+^k[\mathbf{x}] = 1 \cdot \mathbf{u}, \quad \forall k = 2n+1, n \in \mathbb{N}$*;*

(e) $\mathrm{M}_+^k[\mathbf{x}] = e^{\left((k-1)(k-3)\cdots 3\cdot 1\cdot\sigma^k\right)} \cdot \mathbf{u}, \quad \forall k = 2n, n \in \mathbb{N}.$

Proposition 12 *For* $\mathbf{x}_1$*,* $\mathbf{x}_2$*, ...,* $\mathbf{x}_n$ *independent random variables on* $\mathbb{R}_+$*,* $\mathbf{x}_k \sim \mathcal{N}_+(\mu_k, \sigma_k^2)$*,* $k = 1, 2, \ldots, n$*,*

$$\mathbf{s}_n = s_n \cdot \mathbf{u} = \prod_{k=1}^{n} x_k \cdot \mathbf{u} \sim \mathcal{N}_+\left(\sum_{k=1}^{n} \mu_k, \sum_{k=1}^{n} \sigma_k^2\right).$$

Furthermore, for $\mathbf{x}_k$ *independent identically distributed,* $\mathbf{x}_k \sim \mathcal{N}_+(\mu, \sigma^2)$*,* $k = 1, 2, \ldots, n$*,*

$$\mathbf{s}_n \sim \mathcal{N}_+\left(n\mu, n\sigma^2\right) \quad \text{and} \quad \left(\prod_{k=1}^{n} x_k\right)^{1/n} \cdot \mathbf{u} \sim \mathcal{N}_+\left(\mu, \frac{\sigma^2}{n}\right).$$

Finally, for $\mathbf{x}_k \sim \mathcal{N}_+(0,1)$*,* $k = 1, 2, \ldots, n$*,*

$$\left(\prod_{k=1}^{n} x_k\right)^{1/\sqrt{n}} \cdot \mathbf{u} \sim \mathcal{N}_+(0,1).$$

There is nothing special about maximum likelihood estimates of μ and σ^2 in $\mathbb{R}_+$, as they are computed on the coefficients, but they allow us to compute $(1-\gamma)$ confidence intervals around the expected value or center of the distribution $\mathrm{E}_+[\mathbf{x}]$, both for σ known and for σ unknown. In the first case we obtain a shortest length interval in $\mathbb{R}_+$,

$$\left[\exp\left(\hat{\mu} - z_{\gamma/2}\frac{\sigma}{\sqrt{n}}\right)\cdot\mathbf{u}, \exp\left(\hat{\mu} + z_{\gamma/2}\frac{\sigma}{\sqrt{n}}\right)\cdot\mathbf{u}\right],$$

where $z_{\gamma/2}$ stands for the $1-\gamma/2$ quantile of the standard normal distribution in $\mathbb{R}$, and for σ unknown we obtain a shortest expected length in $\mathbb{R}_+$ interval as

$$\left[\exp\left(\hat{\mu} - \mathtt{t}_{n,\gamma/2}\frac{\sigma}{\sqrt{n}}\right)\cdot\mathbf{u}, \exp\left(\hat{\mu} + \mathtt{t}_{n,\gamma/2}\frac{\sigma}{\sqrt{n}}\right)\cdot\mathbf{u}\right],$$

where $\mathtt{t}_{n,\gamma/2}$ stands for the $1-\gamma/2$ quantile of Students $\mathtt{t}$ distribution in $\mathbb{R}$.

Example 2. Let us now look at the case $\mathcal{E} = \mathbb{R}^2_+ \subset \mathbb{R}^2$, which is a two dimensional Euclidean space. Any random vector $\mathbf{x}$ with support space $\mathbb{R}^2_+$ can be expressed as $\mathbf{x} = \ln x_1 \odot \mathbf{w}_1 \oplus \ln x_2 \odot \mathbf{w}_2$, and $[\ln x_1, \ln x_2]$ is a real random vector. The bivariate normal distribution on this space has already been used in (Pawlowsky-Glahn, Egozcue, and Burger, 2003) as a possible model for maximum ocean significant wave height and period.

Definition 9 *A random vector* $\mathbf{x}$ *is said to follow a bivariate normal distribution on* $\mathbb{R}^2_+$ *with parameters* μ_1, μ_2, σ_1, σ_2 *and* ρ *if its density function is*

$$f_{\mathbf{x}}(\mathbf{x}) = f(\ln x_1, \ln x_2) = \frac{1}{2\pi\sigma_1\sigma_2(1-\rho^2)^{1/2}}\exp\left(-\frac{A}{2(1-\rho)^2}\right),$$

where

$$A = \left[\frac{(\ln x_1 - \mu_1)^2}{\sigma_1^2} - \frac{2\rho(\ln x_1 - \mu_1)(\ln x_2 - \mu_2)}{\sigma_1\sigma_2} + \frac{(\ln x_2 - \mu_2)^2}{\sigma_2^2}\right],$$

$0 < x_1$; $0 < x_2$; $\mu_1, \mu_2 \in \mathbb{R}$; $0 < \sigma_1$; $0 < \sigma_2$; $|\rho| < 1$. *We shall write* $\mathbf{x} \sim \mathcal{N}_{\mathbb{R}^{\in}_+}(\mu_1, \mu_2; \sigma_1^2, \sigma_2^2, \rho)$.

Proposition 13 *For* $\mu_1 = \mu_2 = 0$, $\rho = 0$ *and* $\sigma_1 = \sigma_2 = \sigma$, *the components of* $\mathbf{x}$ *are independent.*

In fact, we know that the property is true on the coefficients and, as each component is uniquely associated to a coefficient, the property holds also for the components, as we have

$$f_{\mathbf{x}}(\mathbf{x}) = f(\ln x_1, \ln x_2) = f_+(\ln x_1)\cdot f_+(\ln x_2) = f_{\mathbf{x}_1}(\mathbf{x}_1)\cdot f_{\mathbf{x}_2}(\mathbf{x}_2).$$

Proposition 14 *The moments of a random vector* $\mathbf{x} \sim \mathcal{N}_{\mathbb{R}^{\in}_+}(\mu_1, \mu_2; \sigma_1^2, \sigma_2^2, \rho)$ *satisfy the following properties:*

(a) $\mathrm{E}_{\mathbb{R}^2_+}[\mathbf{x}] = [\exp(\mu_1), \exp(\mu_2)]\cdot\mathbf{U}'$;

(b) $\mathrm{E}_{\mathbb{R}^2_+}\left[\mathbf{x}^2\right] = \left[\exp(\sigma_1^2 + \mu_1^2), \exp(\sigma_2^2 + \mu_2^2)\right]\cdot\mathbf{U}'$;

(c) $\mathrm{M}^2_{\mathbb{R}^2_+}[\mathbf{x}] = \left[\exp(\sigma_1^2), \exp(\sigma_2^2)\right]\cdot\mathbf{U}'$;

(d) $\mathrm{M}^k_{\mathbb{R}^2_+}[\mathbf{x}] = [1,1]\cdot\mathbf{U}', \quad \forall k = 2n+1, n \in \mathbb{N}$;

(e) $\mathrm{M}^k_{\mathbb{R}^2_+}[\mathbf{x}] = \left[\exp\left((k-1)(k-3)\cdots 3\cdot 1\cdot\sigma_1^k\right), \exp\left((k-1)(k-3)\cdots 3\cdot 1\cdot\sigma_2^k\right)\right]\cdot\mathbf{U}'$, $\forall k = 2n, n \in \mathbb{N}$,

where $\mathbf{U} = [\mathbf{u}_1, \mathbf{u}_2]$ *is the canonical basis of* $\mathbb{R}^2$ *in which the observations are measured.*

Note that $\mathrm{E}_{\mathbb{R}^2_+}[\mathbf{x}]$ is the vector of geometric means, thus justifying median based statistics as a reasonable approach for the analysis of bivariate, strictly positive observations. Note also that in this case, the parameter ρ can be clearly interpreted as a measure of *linear* dependence both between the real random coefficients $\ln x_1$ and $\ln x_2$, and between the components.

Example 3. Consider now $\mathcal{E} = \mathcal{I} = (0, 1)$, which is a one dimensional Euclidean space. We start from a given unitary basis $\mathbf{w} = \frac{e}{1+e} \cdot \mathbf{u}$. The expression of a random variable in terms of this basis is

$$\mathbf{x} = \ln \frac{x}{1-x} \odot \mathbf{w} = \frac{\exp\left(\ln \frac{x}{1-x}\right)}{1 + \exp\left(\ln \frac{x}{1-x}\right)} \cdot \mathbf{u} = x \cdot \mathbf{u}.$$

Some aspects related to the geometry of this space have already been studied in (Pawlowsky-Glahn and Egozcue, 2001). Let us now follow the scheme of the general theory stated above in this particular case.

Definition 10 *A random variable* $\mathbf{x}$ *is said to follow a standard normal distribution on* $\mathcal{I} = (0, 1)$ *if its density function is*

$$f_{\mathbf{x}}(\mathbf{x}) = f\left(\ln \frac{x}{1-x}\right) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(\ln \frac{x}{1-x})^2}{2}\right), \quad 0 < x < 1.$$

$\mathbf{x}$ *is said to follow a normal distribution on* $\mathcal{I} = (0, 1)$ *with parameters* μ *and* σ*, if its density function is*

$$f_{\mathbf{x}}(\mathbf{x}) = f\left(\ln \frac{x}{1-x}\right) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(\ln \frac{x}{1-x} - \mu)^2}{2\sigma^2}\right), \quad 0 < x < 1.$$

We shall write for short $\mathbf{x} \sim \mathcal{N}_{\mathcal{I}}(0, 1)$*, respectively* $\mathbf{x} \sim \mathcal{N}_{\mathcal{I}}(\mu, \sigma^2)$*.*

Proposition 15 *If* $\mathbf{x} \sim \mathcal{N}_{\mathcal{I}}(\mu, \sigma^2)$*, then*

$$\left(x^{1/\sigma} \cdot (1-x)^{-1/\sigma} \cdot e^{-\mu/\sigma}\right) \cdot \mathbf{u} \sim \mathcal{N}_{\mathcal{I}}(0, 1).$$

Proposition 16 *The normal distribution on* $\mathcal{I} = (0, 1)$ *is stable.*

Proposition 17 *The moments of a random variable* $\mathbf{x} \sim \mathcal{N}_{\mathcal{I}}(\mu, \sigma^2)$ *satisfy the following properties:*

(a) $\mathrm{E}_+[\mathbf{x}] = \frac{\exp(\mu)}{1+\exp(\mu)} \cdot \mathbf{u}$;

(b) $\mathrm{E}_+[\mathbf{x}^2] = \frac{\exp(\sigma^2+\mu^2)}{1+\exp(\sigma^2+\mu^2)} \cdot \mathbf{u}$;

(c) $\mathrm{M}_+^2[\mathbf{x}] = \frac{\exp(\sigma^2)}{1+\exp(\sigma^2)} \cdot \mathbf{u}$;

(d) $\mathrm{M}_+^k[\mathbf{x}] = \frac{1}{2} \cdot \mathbf{u}, \quad \forall k = 2n+1, n \in \mathbb{N}$;

(e) $\mathrm{M}_+^k[\mathbf{x}] = \frac{\exp\left((k-1)(k-3)\cdots 3 \cdot 1 \cdot \sigma^k\right)}{1+\exp((k-1)(k-3)\cdots 3 \cdot 1 \cdot \sigma^k)} \cdot \mathbf{u}, \quad \forall k = 2n, n \in \mathbb{N}.$

Proposition 18 *For* $\mathbf{x}_1,\ \mathbf{x}_2,\ \ldots,\ \mathbf{x}_n$ *independent random variables on* $\mathcal{I}$, $\mathbf{x}_k \sim \mathcal{N}_\mathcal{I}(\mu_k, \sigma_k^2)$, $k = 1, 2, \ldots, n$,

$$\mathbf{s}_n = s_n \cdot \mathbf{u} = \frac{\prod_{k=1}^n \frac{x_k}{1-x_k}}{1 + \prod_{k=1}^n \frac{x_k}{1-x_k}} \cdot \mathbf{u} \sim \mathcal{N}_\mathcal{I}\left(\sum_{k=1}^n \mu_k, \sum_{k=1}^n \sigma_k^2\right).$$

Furthermore, for $\mathbf{x}_k$ *independent identically distributed,* $\mathbf{x}_k \sim \mathcal{N}_\mathcal{I}(\mu, \sigma^2)$, $k = 1, 2, \ldots, n$,

$$\mathbf{s}_n \sim \mathcal{N}_\mathcal{I}\left(n\mu, n\sigma^2\right) \qquad \text{and} \qquad \frac{\left(\prod_{k=1}^n \frac{x_k}{1-x_k}\right)^{1/n}}{1 + \left(\prod_{k=1}^n \frac{x_k}{1-x_k}\right)^{1/n}} \cdot \mathbf{u} \sim \mathcal{N}_\mathcal{I}\left(\mu, \frac{\sigma^2}{n}\right).$$

Finally, for $\mathbf{x}_k \sim \mathcal{N}_\mathcal{I}(0, 1)$, $k = 1, 2, \ldots, n$,

$$\frac{\left(\prod_{k=1}^n \frac{x_k}{1-x_k}\right)^{1/\sqrt{n}}}{1 + \left(\prod_{k=1}^n \frac{x_k}{1-x_k}\right)^{1/\sqrt{n}}} \cdot \mathbf{u} \sim \mathcal{N}_\mathcal{I}\left(0, 1\right).$$

There is nothing special about maximum likelihood estimates of μ and σ^2 in $\mathcal{I} = (0, 1)$, as they are computed on the coefficients, but they allow us to compute $(1-\gamma)$ confidence intervals around the expected value or center of the distribution $\mathrm{E}_+\left[\mathbf{x}\right]$, both for σ known and for σ unknown. In the first case we obtain a shortest length interval in $\mathcal{I} = (0, 1)$,

$$\left[\frac{\exp\left(\hat{\mu} - z_{\gamma/2}\frac{\sigma}{\sqrt{n}}\right)}{1 + \exp\left(\hat{\mu} - z_{\gamma/2}\frac{\sigma}{\sqrt{n}}\right)} \cdot \mathbf{u}, \frac{\exp\left(\hat{\mu} + z_{\gamma/2}\frac{\sigma}{\sqrt{n}}\right)}{1 + \exp\left(\hat{\mu} + z_{\gamma/2}\frac{\sigma}{\sqrt{n}}\right)} \cdot \mathbf{u}\right],$$

where $z_{\gamma/2}$ stands for the $1-\gamma/2$ quantile of the standard normal distribution in $\mathbb{R}$, while for σ unknown we obtain a shortest expected length in $\mathcal{I} = (0, 1)$ interval as

$$\left[\frac{\exp\left(\hat{\mu} - \mathtt{t}_{n,\gamma/2}\frac{\sigma}{\sqrt{n}}\right)}{1 + \exp\left(\hat{\mu} - \mathtt{t}_{n,\gamma/2}\frac{\sigma}{\sqrt{n}}\right)} \cdot \mathbf{u}, \frac{\exp\left(\hat{\mu} + \mathtt{t}_{n,\gamma/2}\frac{\sigma}{\sqrt{n}}\right)}{1 + \exp\left(\hat{\mu} + \mathtt{t}_{n,\gamma/2}\frac{\sigma}{\sqrt{n}}\right)} \cdot \mathbf{u}\right],$$

where $\mathtt{t}_{n,\gamma/2}$ stands for the $1-\gamma/2$ quantile of Students $\mathtt{t}$ distribution in $\mathbb{R}$.

Example 4. Let us now study the case $\mathcal{E} = (0, 1) \times (0, 1) = \mathcal{I}^2 \subset \mathbb{R}^2$, which is a two dimensional Euclidean space. Any random vector $\mathbf{x}$ with support space $\mathcal{I}^2$ can be expressed as

$$\mathbf{x} = \ln\frac{x_1}{1-x_1} \odot \mathbf{w}_1 \oplus \ln\frac{x_2}{1-x_2} \odot \mathbf{w}_2,$$

and $\left[\ln\frac{x_1}{1-x_1}, \ln\frac{x_2}{1-x_2}\right]$ is a real random vector.

Definition 11 *A random vector* $\mathbf{x}$ *is said to follow a bivariate normal distribution on* $\mathcal{I}^2$ *with parameters* μ_1, μ_2, σ_1, σ_2 *and* ρ *if its density function is*

$$f_\mathbf{x}(\mathbf{x}) = f\left(\ln\frac{x_1}{1-x_1}, \ln\frac{x_2}{1-x_2}\right) = \frac{1}{2\pi\sigma_1\sigma_2(1-\rho^2)^{1/2}} \cdot \exp\left(-\frac{A}{2(1-\rho)^2}\right),$$

where

$$A = \left[\frac{\left(\ln\frac{x_1}{1-x_1} - \mu_1\right)^2}{\sigma_1^2} - \frac{2\rho\left(\ln\frac{x_1}{1-x_1} - \mu_1\right)\left(\ln\frac{x_2}{1-x_2} - \mu_2\right)}{\sigma_1\sigma_2} + \frac{\left(\ln\frac{x_2}{1-x_2} - \mu_2\right)^2}{\sigma_2^2}\right],$$

$0 < x_1 < 1$; $0 < x_2 < 1$; $\mu_1, \mu_2 \in \mathbb{R}$; $0 < \sigma_1$; $0 < \sigma_2$; $|\rho| < 1$. *We shall write for short* $\mathbf{x} \sim \mathcal{N}_{\mathcal{I}^\in}(\mu_1, \mu_2; \sigma_1^2, \sigma_2^2, \rho)$.

Proposition 19 *For $\mu_1 = \mu_2 = 0$, $\rho = 0$ and $\sigma_1 = \sigma_2 = \sigma$, the components of $\mathbf{x}$ are independent.*

In fact, we know that the property is true on the coefficients and, as each component is uniquely associated to a coefficient, the property holds also for the components, as we have

$$f_{\mathbf{x}}(\mathbf{x}) = f\left(\ln \frac{x_1}{1-x_1}, \ln \frac{x_2}{1-x_2}\right) = f_1\left(\ln \frac{x_1}{1-x_1}\right) \cdot f_2\left(\ln \frac{x_2}{1-x_2}\right) = f_{\mathbf{x}_1}(\mathbf{x}_1) \cdot f_{\mathbf{x}_2}(\mathbf{x}_2).$$

Proposition 20 *The moments of a random vector $\mathbf{x} \sim \mathcal{N}_{\mathcal{I}^\in}(\mu_1, \mu_2; \sigma_1^2, \sigma_2^2, \rho)$ satisfy the following properties:*

(a) $\mathrm{E}_{\mathcal{I}^2}[\mathbf{x}] = \left[\frac{\exp(\mu_1)}{1+\exp(\mu_1)}, \frac{\exp(\mu_2)}{1+\exp(\mu_2)}\right] \cdot \mathbf{U}'$;

(b) $\mathrm{E}_{\mathcal{I}^2}\left[\mathbf{x}^2\right] = \left[\frac{\exp(\sigma_1^2+\mu_1^2)}{1+\exp(\sigma_1^2+\mu_1^2)}, \frac{\exp(\sigma_2^2+\mu_2^2)}{1+\exp(\sigma_2^2+\mu_2^2)}\right] \cdot \mathbf{U}'$;

(c) $\mathrm{M}^2_{\mathcal{I}^2}[\mathbf{x}] = \left[\frac{\exp(\sigma_1^2)}{1+\exp(\sigma_1^2)}, \frac{\exp(\sigma_2^2)}{1+\exp(\sigma_2^2)}\right] \cdot \mathbf{U}'$;

(d) $\mathrm{M}^k_{\mathcal{I}^2}[\mathbf{x}] = \left[\frac{1}{2}, \frac{1}{2}\right] \cdot \mathbf{U}', \quad \forall k = 2n+1, n \in \mathbb{N}$;

(e) $\mathrm{M}^k_{\mathcal{I}^2}[\mathbf{x}] = \left[\frac{\exp\left((k-1)(k-3)\cdots 3\cdot 1\cdot \sigma_1^k\right)}{1+\exp\left((k-1)(k-3)\cdots 3\cdot 1\cdot \sigma_1^k\right)}, \frac{\exp\left((k-1)(k-3)\cdots 3\cdot 1\cdot \sigma_2^k\right)}{1+\exp\left((k-1)(k-3)\cdots 3\cdot 1\cdot \sigma_2^k\right)}\right] \cdot \mathbf{U}', \quad \forall k = 2n, n \in \mathbb{N}$,

where $\mathbf{U} = [\mathbf{u}_1, \mathbf{u}_2]$ is the canonical basis of $\mathbb{R}^2$ in which the observations are measured.

In this case, the parameter ρ can be obviously interpreted as a measure of linear dependence both between the real random coefficients and between the components.

Example 5. Let us now study the case $\mathcal{E} = \mathcal{S}^3 \subset \mathbb{R}^3$, which is a two dimensional Euclidean space. Any random vector $\mathbf{x}$ with support space $\mathcal{S}^3$ can be expressed as

$$\mathbf{x} = \left[\frac{1}{\sqrt{2}} \ln \frac{x_1}{x_2}, \frac{1}{\sqrt{6}} \ln \frac{x_1 x_2}{x_3 x_3}\right] \odot \mathbf{W}',$$

and

$$\left[\frac{1}{\sqrt{2}} \ln \frac{x_1}{x_2}, \frac{1}{\sqrt{6}} \ln \frac{x_1 x_2}{x_3 x_3}\right]$$

is a real random vector. The normal distribution on $\mathcal{S}^D$, $D > 1$, has been extensively studied in (Mateu-Figueras, 2003).

Definition 12 *A random vector $\mathbf{x}$ is said to follow a bivariate normal distribution on $\mathcal{S}^3$ with parameters μ_1, μ_2, σ_1, σ_2 and ρ if its density function in the basis $\mathbf{W}$ can be expressed as*

$$f_{\mathbf{x}}(\mathbf{x}) = f\left(\frac{1}{\sqrt{2}} \ln \frac{x_1}{x_2}, \frac{1}{\sqrt{6}} \ln \frac{x_1 x_2}{x_3 x_3}\right) = \frac{1}{2\pi\sigma_1\sigma_2(1-\rho^2)^{1/2}} \cdot \exp\left(-\frac{A}{2(1-\rho)^2}\right),$$

where

$$A = \left[\frac{\left(\frac{1}{\sqrt{2}} \ln \frac{x_1}{x_2} - \mu_1\right)^2}{\sigma_1^2} - \frac{2\rho\left(\frac{1}{\sqrt{2}} \ln \frac{x_1}{x_2} - \mu_1\right)\left(\frac{1}{\sqrt{6}} \ln \frac{x_1 x_2}{x_3 x_3} - \mu_2\right)}{\sigma_1\sigma_2} + \frac{\left(\frac{1}{\sqrt{6}} \ln \frac{x_1 x_2}{x_3 x_3} - \mu_2\right)^2}{\sigma_2^2}\right],$$

$0 < x_1$; $0 < x_2$; $0 < x_3$; $x_1 + x_2 + x_3 = 1$; $\mu_1, \mu_2 \in \mathbb{R}$; $0 < \sigma_1$; $0 < \sigma_2$; $|\rho| < 1$. We shall write for short $\mathbf{x} \sim \mathcal{N}_{\mathcal{S}^\ni}(\mu_1, \mu_2; \sigma_1^2, \sigma_2^2, \rho)$.

Note that the expression of the density function depends on the basis chosen, but at the same time we know that an orthonormal transformation will be enough to move from the expression in one representation to another.

Proposition 21 *For $\mu_1 = \mu_2 = 0$, $\rho = 0$ and $\sigma_1 = \sigma_2 = \sigma$, the coefficients of $\mathbf{x}$ are independent, but this is not true for the parts or components.*

In fact, we know that

$$f_{\mathbf{x}}(\mathbf{x}) = f\left(\frac{1}{\sqrt{2}}\ln\frac{x_1}{x_2}, \frac{1}{\sqrt{6}}\ln\frac{x_1 x_2}{x_3 x_3}\right) = f_1\left(\frac{1}{\sqrt{2}}\ln\frac{x_1}{x_2}\right)\cdot f_2\left(\frac{1}{\sqrt{6}}\ln\frac{x_1 x_2}{x_3 x_3}\right),$$

but this expression cannot be written as the product of three density functions, each associated to one of the parts of the random vector $\mathbf{x}$. Actually, f_1 is a possible expression for the density of the subcomposition $\mathcal{C}\left[x_1 \cdot \mathbf{u}_1 + x_2 \cdot \mathbf{u}_2\right]$. As the only possible concept of independence in the simplex is associated to subcompositions, this particular case corresponds to a degenerate situation (Egozcue, personal communication).

Proposition 22 *The moments of a random vector $\mathbf{x} \sim \mathcal{N}_{\mathcal{S}^{\ni}}(\mu_1, \mu_2; \sigma_1^2, \sigma_2^2, \rho)$ satisfy the following properties:*

(a) $\mathrm{E}_{\mathcal{S}^3}\left[\mathbf{x}\right] = \mathcal{C}\left[\exp\frac{1}{\sqrt{2}}\left(\sqrt{3}\mu_2 + \mu_1\right)\cdot\mathbf{u}_1 + \exp\frac{1}{\sqrt{2}}\left(\sqrt{3}\mu_2 - \mu_1\right)\cdot\mathbf{u}_2 + \mathbf{u}_3\right]$*;*

(b) $\mathrm{M}^2_{\mathcal{S}^3}\left[\mathbf{x}\right] = \mathcal{C}\left[\exp\frac{1}{\sqrt{2}}\left(\sqrt{3}\sigma_2^2 + \sigma_1^2\right)\cdot\mathbf{u}_1 + \exp\frac{1}{\sqrt{2}}\left(\sqrt{3}\sigma_2^2 - \sigma_1^2\right)\cdot\mathbf{u}_2 + \mathbf{u}_3\right]$*;*

(c) $\mathrm{M}^k_{\mathcal{S}^3}\left[\mathbf{x}\right] = \left[\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right]\cdot\mathbf{U}' \quad \forall k = 2n+1, n \in \mathbb{N}$,

where $\mathbf{U} = [\mathbf{u}_1, \mathbf{u}_2]$ is the canonical basis of $\mathbb{R}^2$ in which the observations are measured.

The other two properties related to moments—present in previous examples—have been left out due to their complicated expression. Nevertheless, the fact that $\mathrm{E}_{\mathcal{S}^3}\left[\mathbf{x}\right]$ is the closed vector of medians (Aitchison, 2002) is possibly an indication that it is worthwhile to study them in more detail.

Note that in $\mathcal{S}^3$, the parameter ρ cannot be interpreted as a measure of linear dependence between the components, despite it is certainly one for the coefficients.

6 Conclusions

Statistical analysis on coefficients with respect to an orthonormal basis is not only possible, but also facilitates the study of phenomena which sample space is an arbitrary Euclidean space $\mathcal{E}$. In fact, all standard results valid for real random variables or vectors transfer automatically to the random variables with support $\mathcal{E}$, thus making mathematical proofs trivial and computation easy. Nevertheless, this approach is in general not intuitive at all, at least up to now, and the expression of standard results in terms of the vector space operations in $\mathcal{E}$ is not always easy. Therefore, still a large amount of work is required to make out of the available tools accessible methods for most of the applied scientists which might need them.

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